A bound on the approximation of a maximum matching in a perturbed graph

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The weighted bipartite matching problem can be described by the following LP problem:

$$\text{Maximize } \sum_{ij} c_{ij} x_{ij} \tag{1}$$

Subject to

$$\begin{aligned} \forall i \sum_j x_{ij} &\leq 1, \\ \forall j \sum_i x_{ij} &\leq 1 \\ x_{ij} &\geq 0 \end{aligned} \tag{2}$$

The Dual LP is as follows:

$$\text{Minimize } \sum_i u_i + \sum_j v_j \tag{3}$$

Subject to

$$\begin{aligned} \forall i, j \; u_i + v_j &\geq c_{ij}, \\ u_i, v_i &\geq 0 \end{aligned} \tag{4}$$

The vectors u and v which satisfy inequalities (4) are also called a *covering* in G. The size of a covering, i.e. $\sum_i u_i + \sum_j v_j$ is an upper bound on the weight of any matching in G, with equality if and only if the covering is minimum and the matching is maximum.

We will present an analysis which, given a solution to the optimization problem, allows us to estimate the weight of an optimal solution to a new problem with small perturbations of the edge weight vector.

Assume we are given optimal solutions x^0_{ij} and u^0_i, v^0_i to the LP and the Dual LP problems, respectively, for a given graph G with edge weights vector $\mathbf{c}$. Assume that a new weight vector $\mathbf{c}^1$is given by $c^1_{ij} = c_{ij} + \epsilon_{ij}$, and let x^1_{ij}, and u^1_i, v^1_i be optimal solutions to the new weight vector $\mathbf{c}^1$. A straight forward upper bound to

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the solution of the new problem with perturbed weights can be obtained by defining: $u_i'' = u_i^0 + \max_j \epsilon_{ij}$ and $v_i'' = v_i^0$.

It is not difficult to see that u_i'' and v_i'' is a feasible covering to the new problem, and hence we have the bound

$$\sum_{ij} c_{ij}^1 x_{ij}^1 \leq \sum_i u_i'' + \sum_j v_j'' = \sum_i u_i^0 + \sum_i \max_j \epsilon_{ij} + \sum_i v_i^0 = \sum_{ij} c_{ij} x_{ij}^0 + \sum_i \max_j \epsilon_{ij} \tag{5}$$

Similarly, we can define $u_i'' = u_i^0$ and $v_j'' = v_j^0 + \max_i \epsilon_{ij}$ to obtain another feasible covering to the new problem, and another upper bound to the weight of $\mathbf{x}^1$.

$$\sum_{ij} c_{ij}^1 x_{ij}^1 \leq \sum_i u_i'' + \sum_j v_j'' = \sum_i u_i^0 + \sum_j v_j^0 + \sum_j \max_i \epsilon_{ij} = \sum_{ij} c_{ij} x_{ij}^0 + \sum_j \max_i \epsilon_{ij} \tag{6}$$

The bounds in the RHS of equations 7 and 6 are not necessarily the same and the smaller bound is clearly a better upper bound.

We can obtain better bounds for the optimal matching problem with the perturbed edge weights by improving the coverings u_i'' and v_j'' defined above. One way to do that is by defining

$u_i'' = u_i^0 + \max_j\{\epsilon_{ij} - (u_i + v_j - c_{ij})\}$ and $v_i'' = v_i^0$.

This bound is better than the previous bound since $u_i + v_j - c_{ij} \geq 0$ for all i, j.

The vectors u_i'' and v_j'' are still a feasible covering to the perturbed problem, and we get the improved bound

$$\sum_{ij} c_{ij}^1 x_{ij}^1 \leq \sum_i u_i'' + \sum_j v_j'' = \sum_i u_i^0 + \sum_i \max_j\{\epsilon_{ij} - (u_i + v_j - c_{ij})\} + \sum_i v_i^0 = \sum_{ij} c_{ij} x_{ij}^0 + \sum_i \max_j\{\epsilon_{ij} - (u_i + v_j \tag{7}$$

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Similarly, we can change only the vector v and leave the vector u as it is, i. e. define:

$u_i'' = u_i^0$ and $v_j'' = v_j^0 + \max_i\{\epsilon_{ij} - (u_i + v_j - c_{ij})\}$ to obtain another feasible covering to the new problem.

So far we have chosen an initial vertex covering u_i'' and v_j'' by updating either u or v. Another approach would be to update both the vectors u and v by first finding a minimum vertex cover to the unweighted problem defined by the set of edges in G which are not covered, with the perturbed weights, by u_i^0 and v_i^0. In other words, consider only the edges ij for which $\epsilon_{ij} - (u_i + v_j - c_{ij}) > 0$. We run Konig's algorithm (also known as the "hungarian Algorithm") for finding a maximum cardinality matching and minimum cardinality covering u^1, v^1 for the subgraph spanned by these edges. Now we define

$u_i'' = u_i^0 + \max_j\{\epsilon_{ij} - (u_i + v_j - c_{ij})\}$ and $v_j'' = v_j^0 + \max_i\{\epsilon_{ij} - (u_i + v_j - c_{ij})\}$ only for vertices i, j for which $u_i^1 = 1$ and $v_j^1 = 1$, respectively. For vertices i (or j) which are not in the covering, i.e. $u_i^1 = 0$, (or $v_j^1 = 0$) we define $u_i'' = u_i^0$ (or $v_j'' = v_j^0$).

Once we have defined a feasible covering of the new perturbed graph, we can continue to improve it, or find a new optimal weighted matching. The improvement is done as in the Kuhn-Munkers algorithm [1], only we begin with feasible matching and covering, and we use the equality subgraph to improve the covering and matching. If we run the algorithms till it terminates, we will indeed get an optimal matching and covering of the updated graph. We may be "lucky" when only after relatively few steps, the equality subgraph related to u_i'' and v_j'' defined above, contains a perfect matching. However, this process may take too long, if we are unlucky. Hence, a reasonable heuristics would be to run the algorithm as long as we would like to invest in it, and stop with some covering $\mathbf{u}, \mathbf{v}$ and some matching $\mathbf{M}$. The size of the covering is an upper bound for an optimal matching of the perturbed problem.

References

[1] Harold W. Kuhn, "The Hungarian Method for the assignment problem", Naval Research Logistics Quarterly, 2:8397, 1955. Kuhn's original publication.

[2] Harold W. Kuhn, "Variants of the Hungarian method for assignment problems", Naval Research Logistics Quarterly, 3: 253258, 1956.

[3] J. Munkres, "Algorithms for the Assignment and Transportation Problems", Journal of the Society for Industrial and Applied Mathematics, 5(1):3238, 1957 March.