Consistent and asymptotic-preserving finite-volume domain decomposition methods for singularly perturbed elliptic equations

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1 Introduction

Domain decomposition methods (DDM) using classical transmission conditions that work well for purely elliptic problems can have poor performance when applied to singularly-perturbed equations of advection-diffusion type. To face this challenge, adaptive Dirichlet-Neumann and Robin-Neumann algorithms were introduced in [2], accounting for transport along characteristics. Good convergence properties were also reported in the discrete setting for damped versions [6]. Non-overlapping DDMs of Schwarz-type applied to advection-diffusion equations were analyzed e.g. in [9, 1] and a stabilized finite-element method for singularly perturbed problems is discussed in [8], see also [3, 4] and references therein for heterogeneous couplings.

Our goal is to develop Robin transmission conditions (TCs) such that a finite-volume based non-overlapping DDM is consistent *and* asymptotic-preserving (AP). Consistent here means that, for fixed mesh size, the discrete DDM iterates converge to the discrete solution on the entire domain, and AP means that the singular limit in the DDM yields a convergent limit DDM (for more on AP, see e.g. [7]). We will also show that the continuous DDM satisfies the AP property. In particular, the continuous and discrete DDM are AP when the TC at the outflow boundary vanishes, as already discussed in [8]. Surprisingly and in contrast to the continuous algorithm, the AP property for the discrete DDM can be obtained without the restrictions on the parameters in the TC at the continuous level, see Theorem 3.

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2 The continuous problem and non-overlapping DDM

We consider for $\nu \geq 0$, $a > 0$ and $f \in L^2(-1,1)$ the stationary advection-diffusion equation with homogeneous Dirichlet boundary conditions, i.e.,

$$\mathcal{L}(u) := \nu\partial_{xx}u - a\partial_x u = f \text{ in } \Omega := (-1,1), \quad u(-1) = 0\,, \quad \nu u(1) = 0\,. \tag{1}$$

In the singular limit $\nu = 0$, the PDE in (1) becomes (trivially) advective, and the boundary condition collapses into the inflow condition $u(-1) = 0$ only. It is easy to see that there exists a unique weak solution $u \in H^1(-1,1)$ of (1) for $\nu \geq 0$.

We apply a non-overlapping DDM with two sub-domains $\Omega_1 = (-1,0)$ and $\Omega_2 = (0,1)$ to (1). The problem (1) is then rewritten using at $x = 0$ the Robin TCs

$$\mathcal{B}_1(u) = \nu\partial_x u - au + \lambda u\,, \quad \mathcal{B}_2(u) = -\nu\partial_x u + au + \lambda u\,, \quad \lambda > 0\,. \tag{2}$$

Definition 1 (Continuous DDM)

Let $u_2^0 \in H^1(\Omega_2)$. For $n \in \mathbb{N}$, the n-th (continuous) DDM-iterate $(u_1^n, u_2^n) \in H^1(-1,0) \times H^1(0,1)$ is given as solution of

$$\nu\partial_{xx}u_j^n - a\partial_x u_j^n = f \qquad \text{in} \quad \Omega_j,\ j = 1,2\,, \tag{3}$$

$$u_1^n(-1) = 0\,, \qquad \nu u_2^n(1) = 0\,, \tag{4}$$

$$\nu\mathcal{B}_1(u_1^n) = \nu\mathcal{B}_1(u_2^{n-1})\,, \quad \mathcal{B}_2(u_2^n) = \mathcal{B}_2(u_1^n) \qquad \text{at} \quad x = 0\,. \tag{5}$$

Note that (3)-(5) is equivalent to (1) in the limit $n \to \infty$. In the limit when $\nu \to 0$, we get the stationary advection equation on both sides, and the two Robin TCs (5) degenerate into one Dirichlet TC. Note that the multiplication of $\mathcal{B}_1$ by ν is necessary to remove the TC in the limit $\nu \to 0$. Otherwise, the result in Theorem 1 below for $\nu = 0$ holds iff $\lambda = a$.

The errors $e_j^n := u|_{\Omega_j} - u_j^n$ satisfy (3)-(5) with $f \equiv 0$ due to linearity. Therefore, we have by direct solution

$$e_1^n(x) = A_1^n(\mathrm{e}^{ax/\nu} - \mathrm{e}^{-a/\nu})\,, \qquad e_2^n(x) = A_2^n(1 - \mathrm{e}^{a(x-1)/\nu}) \qquad \text{if } \nu > 0\,,$$
$$e_1^n \equiv 0\,, \qquad e_2^n \equiv A_2^n \qquad \text{if } \nu = 0\,,$$

where $A_1^n, A_2^n \in \mathbb{R}$ satisfy the recurrence relations

$$A_1^n = \frac{-a+\lambda(1-\mathrm{e}^{-a/\nu})}{a\mathrm{e}^{-a/\nu}+\lambda(1-\mathrm{e}^{-a/\nu})}A_2^{n-1}\,, \qquad A_2^n = \frac{-a\mathrm{e}^{-a/\nu}+\lambda(1-\mathrm{e}^{-a/\nu})}{a+\lambda(1-\mathrm{e}^{-a/\nu})}A_1^n \qquad \text{if } \nu > 0\,,$$
$$0(\lambda - a)0 = 0(\lambda - a)A_2^{n-1}\,, \qquad (a+\lambda)A_2^n = (a+\lambda)0 \qquad \text{if } \nu = 0\,.$$

This yields the following convergence result.

Theorem 1 (Convergence and AP property of the continuous DDM)

The sequence of continuous DDM-iterates $\{(u_1^n, u_2^n)\}_{n\in\mathbb{N}}$ *converges pointwise to* $(u|_{\Omega_1}, u|_{\Omega_2})$. *For* $\nu > 0$, *the convergence is linear with convergence factor*

$$\rho = \left|\frac{(a-\lambda)+\lambda e^{-a/\nu}}{(a+\lambda)-\lambda e^{-a/\nu}}\right|\left|\frac{\lambda-(a+\lambda)e^{-a/\nu}}{\lambda+(a-\lambda)e^{-a/\nu}}\right| < 1 \ . \tag{6}$$

Convergence in one iteration is achieved iff $\lambda = \frac{a}{1-e^{-a/\nu}}$ *or in the case* $\nu = 0$.
The continuous DDM (3)-(5) *is AP if* $\lambda = \lambda(\nu)$ *satisfies* $|\lambda - a| = o(1)$ *as* $\nu \to 0$.

3 Cell-centered finite volume discretization

We discretize (1) and (3)-(5) by a cell-centered finite volume method. For given $I \in \mathbb{N}$, let the step-width be $h := 1/I$ and the volumes $V_i := [ih, (i+1)h]$ for $-I \le i < I$ be given. Furthermore, define $f_i := \int_{V_i} f(x)\,dx$. We denote the constant, cell-centered approximation of u in V_i by u_i, and encapsulate these for all V_i in the vector $\mathbf{u} := (u_i)_{i=-I}^{I-1} \in \mathbb{R}^{2I}$. Using centered differences for the diffusion and upwind fluxes for the advection, the discrete version of problem (1) reads

$$\tfrac{\nu}{h}(u_{i-1} - 2u_i + u_{i+1}) + a(u_{i-1} - u_i) = f_i \quad \text{for } -I < i < I-1, \tag{7}$$

$$\tfrac{\nu}{h}(-3u_{-I} + u_{-I+1}) - 2au_{-I} = f_{-I} \ , \tag{8}$$

$$\tfrac{\nu}{h}(u_{I-2} - 3u_{I-1}) + a(u_{I-2} - u_{I-1}) = f_{I-1} \ . \tag{9}$$

Here, we eliminated the ghost values u_{-I-1} and u_I using a linear interpolation of the boundary conditions. Analogously, one obtains the discrete version of (3) and (4), while (5) becomes

$$B_1(\mathbf{u}_1^n) = B_1(\mathbf{u}_2^{n-1}) \ , \qquad B_2(\mathbf{u}_2^n) = B_2(\mathbf{u}_1^n) \ . \tag{10}$$

It remains to discretize the TC (2) to obtain B_1, B_2, and then to eliminate the ghost values $u_{1,0}$ and $u_{2,-1}$. For this, we use centered differences for the diffusion and linear combinations of the values in V_{-1} and V_0 for the other terms to obtain

$$B_1(\mathbf{u}) = \tfrac{\nu}{h}(u_0 - u_{-1}) - a((1-\alpha_1)u_{-1} + \alpha_1 u_0) + \lambda((1-\beta_1)u_{-1} + \beta_1 u_0) \ , \tag{11}$$

$$B_2(\mathbf{u}) = -\tfrac{\nu}{h}(u_0 - u_{-1}) + a((1-\alpha_2)u_{-1} + \alpha_2 u_0) + \lambda((1-\beta_2)u_{-1} + \beta_2 u_0) \ , \tag{12}$$

for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in [0,1]$. Note that $\alpha_j = \beta_j = 0$, $j = 1, 2$, is an upwind discretization, while the centered choice $\alpha_j = \beta_j = 1/2$, $j = 1, 2$, is typically used in the diffusion-dominated case $\nu \gg a$ to obtain second-order convergence in h.

To eliminate the ghost values $u_{1,0}$ and $u_{2,-1}$ in (7), we solve (11) for u_0 and (12) for u_{-1}. To eliminate $u_{2,-1}$ in (11) and $u_{1,0}$ in (12), we solve (7) for $u_{1,0}$ and $u_{2,-1}$. Inserting the resulting expressions and using (10), we obtain the following discrete DDM iteration.

Definition 2 (Discrete DDM)

For given $\mathbf{u}_2^0 \in \mathbb{R}^I$, let $\tilde{B}_1(\mathbf{u}_2^0) := \frac{\nu B_1(\mathbf{u}_2^0)}{\nu - ah\alpha_1 + \lambda h \beta_1}$. For $n \in \mathbb{N}$, the n-th discrete DDM-iterate $(\mathbf{u}_1^n, \mathbf{u}_2^n) \in (\mathbb{R}^I)^2$ satisfies

$$\tfrac{\nu}{h}(u^n_{j,i-1} - 2u^n_{j,i} + u^n_{j,i+1}) + a(u^n_{j,i-1} - u^n_{j,i}) = f_i \ , \tag{13}$$

for $j = 1$, $-I < i < -1$ and for $j = 2$, $0 < i < I - 1$,

$$\tfrac{\nu}{h}(-3u^n_{1,-I} + u^n_{1,-I+1}) - 2au^n_{1,-I} = f_{-I} \ , \tag{14}$$

$$\tfrac{\nu}{h}(u^n_{2,I-2} - 3u^n_{2,I-1}) + a(u^n_{2,I-2} - u^n_{2,I-1}) = f_{I-1} \ , \tag{15}$$

$$\tfrac{\nu}{h}\big(u^n_{1,-2} - 2u^n_{1,-1}\big) + a(u^n_{1,-2} - u^n_{1,-1}) + \tfrac{\nu}{h}c_1 u^n_{1,-1} = f_{-1} - \tilde{B}_1(\mathbf{u}^{n-1}_2) \ , \tag{16}$$

$$\tfrac{\nu}{h}\big(-2u^n_{2,0} + u^n_{2,1}\big) - au^n_{2,0} + \big(\tfrac{\nu}{h} + a\big)c_2 u^n_{2,0} = f_0 - \tilde{B}_2(\mathbf{u}^n_1) \ , \tag{17}$$

where

$$\tilde{B}_1(\mathbf{u}^n_2) = \tfrac{\nu}{h}u^n_{2,0} - \tfrac{\nu}{\nu+ah}c_1\big(f_0 - \tfrac{\nu}{h}(-2u^n_{2,0} + u^n_{2,1}) + au^n_{2,0}\big) \ , \tag{18}$$

$$\tilde{B}_2(\mathbf{u}^n_1) = \big(\tfrac{\nu}{h}+a\big)u^n_{1,-1} - \tfrac{\nu+ah}{\nu}c_2\big(f_{-1} - \tfrac{\nu}{h}(u^n_{1,-2}-2u^n_{1,-1}) - a(u^n_{1,-2}-u^n_{1,-1})\big) \ , \tag{19}$$

$$c_1 = \frac{\frac{\nu}{h}+a(1-\alpha_1)-\lambda(1-\beta_1)}{\frac{\nu}{h}-a\alpha_1+\lambda\beta_1} \ , \quad c_2 = \frac{\frac{\nu}{h}-a\alpha_2-\lambda\beta_2}{\frac{\nu}{h}+a(1-\alpha_2)+\lambda(1-\beta_2)} \ . \tag{20}$$

Note that (13)-(19) is uniquely solvable for all $\nu \geq 0$ iff $c_1 = O(1/\nu)$ and $c_2 = O(\nu)$ as $\nu \to 0$. The resulting system matrix for $\mathbf{u}^n_2$ is weakly chained diagonally dominant, and thus non-singular. The same holds for $\mathbf{u}^n_1$ if $c_1 \leq 1$. Further note that $\tilde{B}_1$ and $\tilde{B}_2$ in (16)-(19) are discrete Robin-to-Dirichlet operators, so that $c_1 = c_2 = 0$ corresponds to Dirichlet TCs, which do not lead to convergence without overlap.

We next investigate how the coefficients α_j, β_j, $j = 1, 2$, must be chosen to obtain a discrete DDM that is consistent with (7)-(9). Since the discretization (13)-(15) is the same as (7)-(9), consistency follows iff the solution to (16)-(19) in the limit when $n \to \infty$ satisfies (7) and vice versa. The solution $\mathbf{u}$ of (7)-(9) solves (16)-(19), as can be directly seen when inserting it into (16)-(19) using (7) for $i = -1, 0$. This only requires that νc_1 and c_2/ν are well-defined for all $\nu \geq 0$ and all $\lambda > 0$. On the other hand, combining (16) and (18) as well as (17) and (19) yields

$$\begin{aligned}
&\tfrac{\nu}{h}(u_{1,-2} - 2u_{1,-1} + u_{2,0}) + a(u_{1,-2} - u_{1,-1}) \\
&\qquad = f_{-1} + \tfrac{\nu}{\nu+ah}c_1\big(f_0 - \tfrac{\nu}{h}(u_{1,-1} - 2u_{2,0} + u_{2,1}) - a(u_{1,-1} - u_{2,0})\big) \ , \\
&\tfrac{\nu}{h}(u_{1,-1} - 2u_{2,0} + u_{2,1}) + a(u_{1,-1} - u_{2,0}) \\
&\qquad = f_0 + \tfrac{\nu+ah}{\nu}c_2\big(f_{-1} - \tfrac{\nu}{h}(u_{1,-2} - 2u_{1,-1} + u_{2,0}) - a(u_{1,-2} - u_{1,-1})\big) \ .
\end{aligned}$$

Inserting the left-hand sides into the right-hand sides of the other equation, we obtain equivalence with (7) iff $1 \neq c_1 c_2$. Hence, we have proved the following Theorem which provides choices for the TC parameters $\alpha_1, \alpha_2, \beta_1, \beta_2$ that ensure consistency for all $\lambda > 0$ and $\nu \geq 0$.

Theorem 2 (Consistency of the discrete DDM)

The limit of the discrete DDM iterates (13)-(19) *as* $n \to \infty$ *is equal to the solution* $\mathbf{u}$ *of* (7)-(9) *for all* $\lambda > 0$ *if the following conditions hold:*

(A1) $\alpha_1 < \frac{\nu}{ah}$ *(or equal if* $\beta_1 > 0$*), and*

(A2) $\nu c_1 = O(1)$ *as* $\nu \to 0$, *i.e. by* (A1)*,* $\nu = O(\nu - ah\alpha_1 + \lambda h\beta_1)$, *and*
(A3) $1 = O(2 - \alpha_2 - \beta_2)$, *and*
(A4) $c_2 = O(\nu)$ *as* $\nu \to 0$, *i.e. by* (A3)*,* $\alpha_2 + \beta_2 = O(\nu)$, *and*
(A5) $c_1 c_2 \neq 1$*, i.e.,*

$$0 \neq a^2(\alpha_2 - \alpha_1) + \lambda\big(\tfrac{2\nu}{h} + a(\beta_1 + \beta_2 - \alpha_1 - \alpha_2)\big) + \lambda^2(\beta_1 - \beta_2) \ .$$

Remark 1 Note that the simplest choice of the coefficients, which satisfies Theorem 2 is $\alpha_1 = \alpha_2 = \beta_2 = 0$ and $\beta_1 = 1/2$. As shown below, this also yields convergence for any positive discrete Peclet number $\mathrm{Pe} := ah/\nu > 0$. Furthermore, this choice ensures that the discrete DDM is AP as $\nu \to 0$ for any $\lambda > 0$, as we show next.

We split the convergence analysis of the discrete DDM given in definition 2 into two regimes due to the different types of solutions: the elliptic case $\nu > 0$ and the singular limit $\nu = 0$. For this, let $\mathbf{e}^n := \mathbf{u} - (\mathbf{u}_1^n, \mathbf{u}_2^n)$ be the error of the discrete DDM at iteration n. By linearity, $\mathbf{e}^n$ satisfies the discrete DDM (13)-(19) with $\mathbf{f} = \mathbf{0}$.

The elliptic case $\nu > 0$ **:** Then, (13)-(15) for $\mathbf{e}^n$ yield the solution

$$\mathbf{e}^n = \Big(A_1^n\big(\xi^{(i+1)h} - \big(1 + \tfrac{\mathrm{Pe}}{2}\big)\xi^{-1}\big)_{i=-I}^{-1} \ , \ A_2^n\big(1 + \tfrac{\mathrm{Pe}}{2} - \xi^{(i+1)h-1}\big)_{i=0}^{I-1}\Big) \ ,$$

where we defined $\xi := (1 + \mathrm{Pe})^I$. The constants $A_1^n, A_2^n \in \mathbb{R}$ are determined by (16)-(19), which yield the recurrence relations

$$A_1^n = -\frac{\lambda - a + \big(a\alpha_1 - \lambda(\mathrm{Pe}^{-1} + \beta_1)\big)\frac{2\mathrm{Pe}}{2+\mathrm{Pe}}\xi^{-1}}{\big(a\alpha_1 - \lambda(\mathrm{Pe}^{-1} + \beta_1)\big)\frac{2\mathrm{Pe}}{2+\mathrm{Pe}} + (\lambda - a)\xi^{-1}} A_2^{n-1}, \ \ A_2^n = \frac{a\alpha_2 + \lambda(\mathrm{Pe}^{-1} + \beta_2) - (\lambda + a)\frac{2+\mathrm{Pe}}{2\mathrm{Pe}}\xi^{-1}}{(\lambda + a)\frac{2+\mathrm{Pe}}{2\mathrm{Pe}} - \big(a\alpha_2 + \lambda(\mathrm{Pe}^{-1} + \beta_2)\big)\xi^{-1}} A_1^n \ .$$

Therefore, the iteration is linearly convergent iff

$$\rho = \left|\frac{\lambda - a + \big(a\alpha_1 - \lambda(\mathrm{Pe}^{-1} + \beta_1)\big)\frac{2\mathrm{Pe}}{2+\mathrm{Pe}}\xi^{-1}}{\lambda + a - \big(a\alpha_2 + \lambda(\mathrm{Pe}^{-1} + \beta_2)\big)\frac{2\mathrm{Pe}}{2+\mathrm{Pe}}\xi^{-1}}\right| \left|\frac{a\alpha_2 + \lambda(\mathrm{Pe}^{-1} + \beta_2) - (\lambda + a)\frac{2+\mathrm{Pe}}{2\mathrm{Pe}}\xi^{-1}}{a\alpha_1 - \lambda(\mathrm{Pe}^{-1} + \beta_1) + (\lambda - a)\frac{2+\mathrm{Pe}}{2\mathrm{Pe}}\xi^{-1}}\right| < 1 \ . \quad (21)$$

Note that convergence in one iteration is possible for the choice

$$\lambda = \lambda_{\mathrm{opt}} := \frac{2\nu + ah - 2\alpha_1 ah\xi^{-1}}{2\nu + ah - 2\,(\nu + \beta_1 ah)\,\xi^{-1}} a \quad \overset{h\to 0}{\longrightarrow} \quad \frac{a}{1 - e^{-a/\nu}} \ , \qquad (22)$$

which is almost mesh independent when $\alpha_1 = 0$ and $\beta_1 = 1/2$. This is consistent with the continuous DDM and also yields $\lambda_{\mathrm{opt}} \to a$ as $\nu \to 0$.

Furthermore, note that (21) for $\alpha_1 = \alpha_2 = 0$ and $\beta_1 = \beta_2 = 1/2$ is satisfied for all $\lambda > 0$. But $\beta_2 = 1/2$ does not satisfy (A4) of Theorem 2, so that $\tilde{B}_2$ (and thus ρ) degenerate when $\nu \to 0$. However, choosing $\alpha_1 = \alpha_2 = \beta_2 = 0$ and $\beta_1 = 1/2$, Theorem 2 is satisfied for all $\nu > 0$, and (21) simplifies to the condition

$$\big|\lambda(1 - \xi^{-1}) - a\big|\,\big|\lambda\big(\tfrac{2}{2+\mathrm{Pe}} - \xi^{-1}\big) - a\xi^{-1}\big| < \big(\lambda\big(1 - \tfrac{2}{2+\mathrm{Pe}}\xi^{-1}\big) + a\big)\big(\lambda(1 - \xi^{-1}) + a\xi^{-1}\big) \ ,$$

which is satisfied for all $\lambda > 0$ due to $\mathrm{Pe} > 0$.

The singular limit $\nu = 0$: Then, (13)-(15) for $\mathbf{e}^n$ yields

$$\mathbf{e}^n = \left((0)_{i=-I}^{-2}, A_1^n, (A_2^n)_{i=0}^{I-1} \right) ,$$

with $A_1^n, A_2^n \in \mathbb{R}$ determined by (16)-(19). To obtain $A_1^1 = 0$, i.e., the correct solution in Ω_1, this requires by (16)

$$0 = A_1^1 = \frac{-\tilde{B}_1(\mathbf{e}^0)}{\frac{\nu}{h}c_1 - a} , \quad \tilde{B}_1(\mathbf{e}_2^0) = \frac{\nu B_1(\mathbf{e}_2^0)}{\nu - ah\alpha_1 + \lambda h\beta_1} .$$

Since $\nu c_1 = O(1)$ as $\nu \to 0$ by (A2), this holds iff $\lim_{\nu\to 0} \nu c_1 \neq ah$ and $\lim_{\nu\to 0} \nu/(\nu - ah\alpha_1 + \lambda h\beta_1) = 0$. Using (A1), this simplifies to $\nu/\beta_1 = o(1)$ as $\nu \to 0$ and implies $c_1 = o(1)$. For A_2^1, we then obtain by (17)-(19)

$$a(c_2 - 1)A_2^1 = -a\left(1 - \frac{\nu + ah}{\nu}c_2\right) A_1^1 .$$

By (A4) of Theorem 2, this yields $A_2^1 = 0$, i.e., convergence in one iteration. Then, $A_1^n = A_2^n = 0$ for all $n > 2$ follows by induction using (16)-(19).

Summarizing the above analysis, we obtain the following result.

Theorem 3 (Convergence and AP property of the discrete DDM)

Let (A1)-(A5) *from Theorem 2 be satisfied. The sequence of discrete DDM iterates* $\{(\mathbf{u}_1^n, \mathbf{u}_2^n)\}_{n\in\mathbb{N}}$ *from* (13)-(19) *converges linearly to the solution of* (7)-(9) *for* $\nu > 0$ *iff* (21) *is satisfied.*
Convergence in one iteration is achieved if λ *satisfies* (22) *or for* $\nu = 0$ *if the limit discrete DDM for* $\nu/\beta_1 = o(1)$ *as* $\nu \to 0$ *is used.*
The discrete DDM (13)-(19) *is AP if* $|\lambda - a| = o(1)$ *or* $\nu/\beta_1 = o(1)$ *as* $\nu \to 0$.

Note that as shown above, the choice $\alpha_1 = \alpha_2 = 0$ and $\beta_1 = \beta_2 = 1/2$ yields linear convergence for $\nu > 0$, but the convergence rate degenerates for $\nu \to 0$. The choice $\alpha_1 = \alpha_2 = \beta_2 = 0$ and $\beta_1 = 1/2$ leads to linear convergence for $\nu > 0$ uniformly in ν with 1-step convergence for $\nu = 0$, and thus is AP.

4 Numerical example

We now study numerically the convergence properties of the discrete DDM as $\nu \to 0$ for various choices of the parameters in the discrete TCs. Since $\alpha_j = O(\nu)$, $j = 1, 2$, is required for convergence, we restrict our study to $\alpha_1 = \alpha_2 = 0$ and vary only β_1, β_2 and λ.

We consider (1) for $f(x) = -\nu(k\pi)^2 \sin(k\pi x) - ak\pi \cos(k\pi x)$, which leads to the exact solution $u(x) = \sin(k\pi x)$. We fix $a = 1$, $k = 3$, $B_1(u_2^0) = 1$ and $I = 100$, and study the number of iterations required to reach an error of $\|\mathbf{e}^n\|_\infty < 10^{-12}$, see Fig. 1. As discussed above, the choice $\beta_1 = \beta_2 = 1/2$ leads to a degeneration as $\nu \to 0$, while the choice $\beta_1 = \beta_2 = \min(1/2, \nu/(ah))$ yields linear convergence,

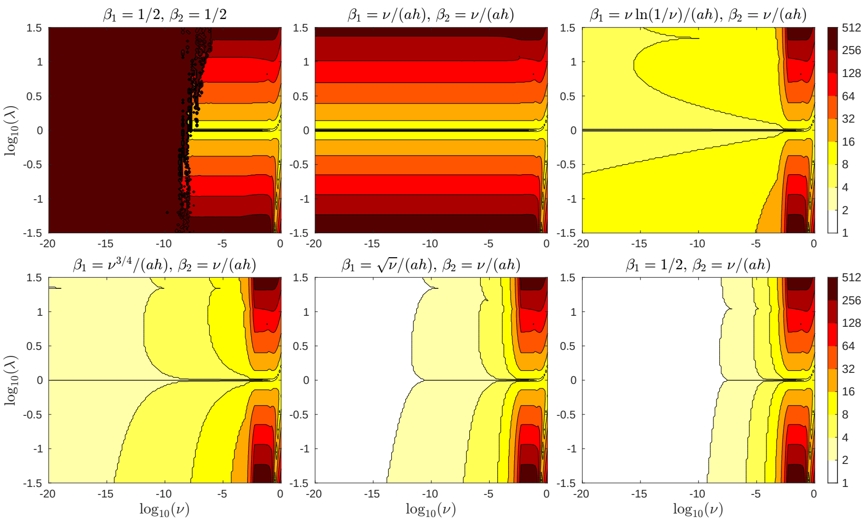

Fig. 1 Number of iterations for various choices of β_1 and β_2.

but is only asymptotic preserving for $\lambda \to a$. As predicted by Theorem 3, the convergence improves for all choices such that $\nu/\beta_1 = o(1)$ and $\beta_2 = O(\nu)$ as $\nu \to 0$. In particular, the number of iterations decreases faster when β_1 is large, which illustrates well the convergence factor ρ in (21), which satisfies

$$\rho = \frac{|\lambda - a|}{\lambda + a} O\left(\frac{\nu}{\nu + \beta_1}\right) + O\left(\nu^{I-1}\right).$$

5 Conclusion

The continuous non-overlapping DDM with Robin TC applied to singularly-perturbed advection-diffusion problems is asymptotic preserving only when the transmission parameter λ tends to the advection speed as $\nu \to 0$. A discrete DDM based on a cell-centered finite volume method can inherit this property. In fact, the discretization of the TC even permits an improved convergence behavior. In contrast to the continuous algorithm, a proper, but asymmetric choice of the discrete parameters (α_j, β_j, $j = 1, 2$) yields the AP property without any restriction on the transmission parameter λ, see Theorem 3. Finally, we illustrated the theoretical results by a numerical example. In the forthcoming work [5] we will exploit our findings to construct a robust DDM for nonlinear convection-diffusion equations.

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