Error estimates for a mixed finite element discretization of a two-phase porous media flow model with dynamic capillarity

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Abstract We analyse a fully discrete numerical scheme for the model describing two-phase immiscible flow in porous media with dynamic effects in the capillary pressure. We employ the Euler implicit method for the time discretization. The spatial discretization is based on the mixed finite element method (MFEM). Specifically, the lowest order Raviart-Thomas elements are applied. In this paper, the error estimates for the saturation, fluxes and phase pressures in $L^\infty(0,T;L^2(\Omega))$ are derived for the temporal and spatial discretization to show the convergence of the scheme. Finally, we present some numerical results to support the theoretical findings.

Keywords Dynamic capillary pressure, Immiscible two-phase flow, Mixed finite elements, Euler implicit method

1 Introduction

In this paper we consider the following model which is used to describe two-phase flow through porous media incorporating dynamic capillarity effects ([33]):

$$\partial_t s - \nabla \cdot (k_o(s)\nabla \bar{p}) = 0, \tag{1}$$

$$-\partial_t s - \nabla \cdot (k_w(s)\nabla p) = 0, \tag{2}$$

$$\bar{p} - p = p_c(s) + \tau \partial_t s, \tag{3}$$

complemented with suitable initial and boundary conditions. The equations hold in $Q := (0,T] \times \Omega$. Here Ω is a bounded domain in $\mathbb{R}^d$ $(d \geq 1)$, having Lipschitz continuous boundary, and $T > 0$ is a given maximal time. The unknowns are s, $\bar{p}$ and p. In order to close the above system, we prescribe the initial and boundary conditions

$$s(0,\cdot) = s^0, \qquad \text{in} \quad \Omega, \tag{4}$$

$$\bar{p} = p = 0, \qquad \text{at} \quad \partial\Omega \quad \text{for} \quad t > 0, \tag{5}$$

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where s^0 is a given function, which will be specified later.

The system (1) - (5) is obtained by including Darcy's law in the mass conservation laws. The mass conservation equations for the two phases are (see [5,34]):

$$\phi\frac{\partial s}{\partial t} + \nabla\cdot\bar{\mathbf{q}} = 0, \tag{6}$$

$$\phi\frac{\partial(1-s)}{\partial t} + \nabla\cdot\mathbf{q} = 0. \tag{7}$$

The coefficient ϕ represents the porosity of the medium, assumed to be constant in this case, s denotes the nonwetting saturation, while $\bar{\mathbf{q}}$ and $\mathbf{q}$ denote the volumetric velocities of nonwetting (e.g. oil) and wetting phases (e.g. water). The volumetric velocities $\bar{\mathbf{q}}$ and $\mathbf{q}$ are deduced from the Darcy's law (see [45]) as

$$\bar{\mathbf{q}} = -\frac{\bar{k}}{\mu_o}k_{ro}(s)\nabla\bar{p}, \tag{8}$$

.

$$\mathbf{q} = -\frac{\bar{k}}{\mu_w}k_{rw}(s)\nabla p, \tag{9}$$

where $\bar{k}$ is the absolute permeability of the porous medium, $\bar{p}$ and p are the phase pressures, μ_α ($\alpha\in\{w,o\}$) the viscosities and $k_{r\alpha}$ ($\alpha\in\{w,o\}$) the relative permeabilities. The specific forms of the functions μ_α and $k_{r\alpha}$ ($\alpha\in\{w,o\}$) are assumed to be known. Substituting (8) and (9) in (6) and (7) gives

$$\phi\frac{\partial s}{\partial t} - \nabla\cdot\left(\frac{\bar{k}k_{ro}}{\mu_o}\nabla\bar{p}\right) = 0, \tag{10}$$

$$-\phi\frac{\partial s}{\partial t} - \nabla\cdot\left(\frac{\bar{k}k_{rw}}{\mu_w}\nabla p\right) = 0. \tag{11}$$

To close the model, we need a constitutive equation that relates the phase pressures $\bar{p}, p$ and the nonwetting saturation s. Motivated by the experimental results in [6,16], Hassanizadeh and Gray (see [33]) derived the following relation:

$$\bar{p} - p = p_c(s) + \tau\frac{\partial s}{\partial t}, \tag{12}$$

where τ denotes the dynamic capillary coefficient and p_c represents the capillary pressure under equilibrium conditions. The phase mobilities are given by

$$k_o(s) := \frac{k_{ro}}{\mu_o}, \qquad k_w(s) := \frac{k_{rw}}{\mu_w}.$$

The system (1) - (5) is derived from (10) - (12) by proper non-dimensionalization [42]. The gravity term is not considered for the mathematical analysis but is included in the numerical section.

Alternatively, if only equilibrium case is considered, the algebraic relationship between phase pressures and saturation is written as

$$\bar{p} - p = p_c(s),$$

which is well known among standard models in porous media (see [5,34,40]). Numerical methods for the standard models have been the subject of extensive research in the last decades. We refer to [4,8,9,19,24,28,46,50,51,53], where finite element method, mixed finite element method, discontinuous Galerkin method are analyzed, or linear iterative schemes are investigated. In [10, 29,43], the authors analyze finite volume methods. The major challenge in developing efficient numerical schemes is related to the degenerate nature of the problem. Due to this, the solution typically lacks regularity, which makes lower order finite elements or finite volumes a natural choice for the spatial discretization. In all cases, the convergence of the numerical schemes is proved rigorously either by compactness arguments, or by obtaining a-priori error estimates. A-posteriori error estimates are obtained e.g. in [10].

Under assuming that the total flow is known, the model can be reduced into a scalar model. The existence and uniqueness of a weak solution for the model including the dynamic capillarity model with $\tau > 0$, is obtained in [11,30,41,44]. A travelling wave analysis has been given in [22, 23]. Numerical schemes for two phase flow through heterogeneous media are discussed in [35] for cases without an entry pressure. For situations including an entry pressure, variational inequality approaches have been considered in [36]. Further, we refer to [21] for coupling conditions between heterogeneous blocks under the dynamic effect. In [35,47], the authors consider numerical algorithms for unsaturated flow through highly heterogeneous media with dynamic effect. For the full two-phase flow model, the existence and uniqueness of the weak solutions are proved in [39,12], with the assumption that the equations are non-degenerate (i.e. all non-linearities are bounded away from 0 or $+\infty$). For this case, a finite volume-finite element method and two point flux approximation are proposed in [17,18], a multipoint flux approximation finite volume method is presented in [14] and a discontinuous Galerkin scheme is proposed in [37,38]. In addition, numerical investigations for heterogeneous media have been given in [32]. For the degenerate case, we refer to [13], which proves the existence of weak solutions for the model using an equivalent formulation.

The rest of the paper is organized as follows. In Section 2, we present the notations and assumptions on the data and the definition of the weak solution. We introduce the mixed formulation and give error estimates for the saturation, phase pressures and fluxes in $L^\infty(0,T;L^2(\Omega))$ for the mixed finite element scheme in Section 3. In the last section, we present some numerical results that confirm the theoretically obtained analysis.

2 Notations and assumptions

In what follows, we use the standard notations of functional analysis and the theory of partial differential equations. Throughout this paper, we assume the system is defined in a bounded connected domain $\Omega \in \mathbb{R}^d (d \geq 1)$. For simplicity, one can assume that Ω is polygonal. Furthermore, by $(\cdot,\cdot)$, we denote the inner product on $L^2(\Omega)$, and let $\|\cdot\|$, $\|\cdot\|_1$ stand for the norms in $L^2(\Omega)$ and $W^{1,2}(\Omega)$, respectively. The functions in $H(\text{div};\Omega)$ are vector valued, having a L^2 divergence. By C we mean a generic positive constant, not depending on the unknowns or the discretization parameters.

For the spatial discretization, let $\mathcal{T}_h$ be a regular decomposition of $\Omega \in \mathbb{R}^d$ into closed $d-$simplices; h denoting the mesh diameter of the decomposition. Here we assume $\bar{\Omega} = \cup_{K\in\mathcal{T}_h} \bar{K}$. Correspondingly, the discrete subspaces $W_h \subset L^2(\Omega)$ and $V_h \subset H(\text{div};\Omega)$ are defined as:

$$W_h := \{p \in L^2(\Omega) | \quad p \text{ is constant on each element } T \in \mathcal{T}_h\},$$
$$V_h := \{\mathbf{q} \in H(\text{div};\Omega) | \quad \mathbf{q}|_K(\mathbf{x}) = \mathbf{a} + b\mathbf{x}, \mathbf{a} \in \mathbb{R}^d,\ b \in \mathbb{R} \text{ for all } K \in \mathcal{T}_h\}.$$

In the following, we use the usual L^2 projector:

$$P_h : L^2(\Omega) \to W_h \text{ such that } (P_h w - w, w_h) = 0, \tag{13}$$

for all $w_h \in W_h$. Furthermore, on $(W^{1,2}(\Omega))^d$ a projector Π_h can be defined such that

$$\Pi_h : (W^{1,2}(\Omega))^d \to V_h,\ (\nabla \cdot (\Pi_h \mathbf{v} - \mathbf{v}), w_h) = 0, \tag{14}$$

for all $w_h \in W_h$. Following [49], pp.237, this operator can be extended to $H(\text{div};\Omega)$. For the above operators there holds

$$\|w - P_h w\| \leq Ch\|w\|_1, \tag{15}$$

$$\|\mathbf{v} - \Pi_h \mathbf{v}\| \leq Ch\|\mathbf{v}\|_1, \tag{16}$$

for any $w \in W^{1,2}(\Omega)$ and $\mathbf{v} \in (W^{1,2}(\Omega))^d$.

For the time discretization, let $N \in \mathbb{N}$ be strictly positive. Then we define the time step $\Delta t = T/N$, as well as $t^n = n\Delta t$ $(n \in \{1, 2, ..., N\})$.

Throughout this paper, we use the following assumptions:

- **(A1)** Ω is an open, bounded and connected, convex polygonal domain in $\mathbb{R}^d$ $(d \geq 1)$ with Lipschitz continuous boundary $\partial\Omega$. $\bar{\Omega}$ denotes the closure of Ω.
- **(A2)** The functions $k_o(\cdot)$, $k_w(\cdot)$: $\mathbb{R} \to \mathbb{R}$ are C^1 and Lipschitz continuous. There exist $0 < \delta_o, \delta_w, M_o, M_w < \infty$ such that $\delta_o \leq k_o(v) \leq M_o$ and $\delta_w \leq k_w(v) \leq M_w$ for all $v \in \mathbb{R}$. We assume $k_o(\cdot)$ to be a non-decreasing function with $k_o(v) = \delta_o$ for $v \leq 0$ and $k_o(v) = M_o$ for $v \geq 1$. $k_w(\cdot)$ is non-increasing with $k_w(v) = M_w$ for $v \leq 0$ and $k_w(v) = \delta_w$ for $v \geq 1$.
- **(A3)** $p_c(\cdot) : \mathbb{R} \to \mathbb{R}$ is an increasing function and $p_c \in C^1$. There exist $m_p, M_p > 0$ such that $m_p \leq p_c^{'}(\cdot) \leq M_p < \infty$.
- **(A4)** $\tau > 0$ is a positive constant.
- **(A5)** The initial condition s^0 in $W_0^{1,2}(\Omega)$ and satisfies $0 \leq s^0 \leq 1$ a.e. .
- **(A6)** The wetting and nonwetting fluxes $\bar{\mathbf{q}}$ and $\mathbf{q}$ are bounded in $L^\infty((0,T)\times\Omega)$, $\|\bar{\mathbf{q}}\|_{L^\infty((0,T)\times\Omega)}$ $+ \|\mathbf{q}\|_{L^\infty((0,T)\times\Omega)} \leq C$.

Remark 1 *Here the assumption* **(A6)** *is reasonable if Ω is a $C^{1,\gamma}$ domain with $0 < \gamma \leq 1$and if $s_0 \in C^{0,\gamma}(\Omega)$. The justification can be found in [11, 13].*

In the following, we define a weak solution of system (1) - (5).

Problem P Find $(s, \bar{p}, p) \in W^{1,2}(0,T; L^2(\Omega)) \times L^2(0,T; W_0^{1,2}(\Omega)) \times L^2(0,T; W_0^{1,2}(\Omega))$, with $s|_{t=0} = s^0$ such that for a.e. $t \in [0,T]$, one has

$$(\partial_t s, \phi) + (k_o(s)\nabla\bar{p}, \nabla\phi) = 0, \tag{17}$$

$$-(\partial_t s, \psi) + (k_w(s)\nabla p, \nabla\psi) = 0, \tag{18}$$

$$(\bar{p} - p, \lambda) = (p_c(s), \lambda) + \tau(\partial_t s, \lambda), \tag{19}$$

for all $\phi \in W_0^{1,2}(\Omega)$, $\psi \in W_0^{1,2}(\Omega)$ and $\lambda \in L^2(\Omega)$.

Remark 2 *In [41], the authors obtain higher regularity for the solution: $\partial_{tt}s \in L^2(0,T; L^2(\Omega))$, and $\bar{p}, p \in W^{1,2}(0,T; W^{1,2}(\Omega))$. In [12], it is also showed $s \in W^{1,2}(0,T; W^{1,2}(\Omega))$.*

Remark 3 (Generality of the analysis) *For the sake of simplicity, we have assumed homogeneous Dirichlet conditions at the boundary and neglected gravity and source terms in* (1) *and* (2) *while analysing the problem in Section 3. However, the results obtained, hold for general Dirichlet, Neumann, Robin and mixed type boundary conditions and they are valid even if gravity terms and linear source terms are included. To show this, the numerical results are given for more general boundary conditions are considered and gravity and source terms are added* (*see* (75)-(79)).

3 Mixed formulation and error estimates

3.1 Existence and uniqueness

We now present a continuous mixed variational formulation for the model (1) - (5).
Problem 1 (*Continuous Variational Formulation*). Find $(s_e, \bar{\mathbf{q}}_e, \bar{p}_e, \mathbf{q}_e, p_e) \in W^{1,2}(0,T;L^2(\Omega)) \times L^2(0,T;H(\mathrm{div};\Omega)) \times L^2(0,T;L^2(\Omega)) \times L^2(0,T;H(\mathrm{div};\Omega)) \times L^2(0,T;L^2(\Omega))$, with $s_e|_{t=0} = s^0$ such that

$$(\partial_t s_e, w) + (\nabla\cdot\bar{\mathbf{q}}_e, w) = 0, \tag{20}$$

$$(k_o^{-1}(s_e)\bar{\mathbf{q}}_e, \mathbf{v}) - (\bar{p}_e, \nabla\cdot\mathbf{v}) = 0, \tag{21}$$

$$-(\partial_t s_e, w) + (\nabla\cdot\mathbf{q}_e, w) = 0, \tag{22}$$

$$(k_w^{-1}(s_e)\mathbf{q}_e, \mathbf{v}) - (p_e, \nabla\cdot\mathbf{v}) = 0, \tag{23}$$

$$(\bar{p}_e - p_e, w) = (p_c(s_e), w) + \tau(\partial_t s_e, w), \tag{24}$$

for all $w \in L^2(\Omega)$ and $\mathbf{v} \in H(\mathrm{div};\Omega)$.

In the following, we discuss the existence and uniqueness of the continuous variational formulations for the system (20) - (24). Since the existence and uniqueness of the model (17) - (19) have been proved in [11–13,44], we follow the approach in [31,51,52] and show the equivalence of Problem P and Problem 1.

Proposition 1 *Let $s \in W^{1,2}(0,T;L^2(\Omega))$ and $\bar{p}, p \in L^2(0,T;W_0^{1,2}(\Omega))$ solve Problem P. Define $(s_e, \bar{p}_e, p_e) := (s, \bar{p}, p)$, $\bar{\mathbf{q}}_e = -k_o(s_e)\nabla\bar{p}_e$ and $\mathbf{q}_e = -k_w(s_e)\nabla p_e$. Then $(s_e, \bar{\mathbf{q}}_e, \bar{p}_e, \mathbf{q}_e, p_e)$ solves Problem 1. Conversely, if $(s_e, \bar{\mathbf{q}}_e, \bar{p}_e, \mathbf{q}_e, p_e)$ solves Problem 1, then $(s, \bar{p}, p) := (s_e, \bar{p}_e, p_e)$ is the solution of Problem P.*

Proof " $\Rightarrow$ " By the definition of $(s_e, \bar{p}_e, p_e)$, it is easy to see that $(s_e, \bar{p}_e, p_e)$ has the regularity required for solving Problem P: $s_e \in W^{1,2}(0,T;L^2(\Omega))$, and $\bar{p}_e, p_e \in L^2(0,T;W_0^{1,2}(\Omega))$. So, we have $\partial_t s_e \in L^2(0,T;L^2(\Omega))$. Then using (17) and by the definition of $\bar{\mathbf{q}}_e$, one obtains

$$(\partial_t s_e, \phi) - (\bar{\mathbf{q}}_e, \nabla\phi) = 0,$$

for any $\phi \in W_0^{1,2}(\Omega)$, which implies that $\nabla\cdot\bar{\mathbf{q}}_e = -\partial_t s_e$ in distributional sense. The regularity of $\partial_t s_e$ immediately implies $\bar{\mathbf{q}}_e \in H(\mathrm{div};\Omega)$ which is required for Problem 1. Similarly, we also have $\mathbf{q}_e \in H(\mathrm{div};\Omega)$.

" $\Leftarrow$ " Clearly, if $(s_e, \bar{\mathbf{q}}_e, \bar{p}_e, \mathbf{q}_e, p_e)$ solves Problem 1, then we have $s_e \in W^{1,2}(0,T;L^2(\Omega))$ and $\bar{p}_e \in L^2(0,T;L^2(\Omega))$, which imply $s \in W^{1,2}(0,T;L^2(\Omega))$ and $\bar{p} \in L^2(0,T;L^2(\Omega))$. Taking $\mathbf{v} \in (C_0^\infty(\Omega))^d \subset H(\mathrm{div};\Omega)$ in (21), one has

$$(\nabla\bar{p}_e, \mathbf{v}) = -(k_o^{-1}(s_e)\bar{\mathbf{q}}_e, \mathbf{v}), \tag{25}$$

which gives $\nabla\bar{p}_e = -k_o^{-1}(s_e)\bar{\mathbf{q}}_e$ in distributional sense. Further, since $s_e \in W^{1,2}(0,T;L^2(\Omega))$ and $\bar{\mathbf{q}}_e \in L^2(0,T;H(\mathrm{div}),\Omega)$, one obtains $\bar{p}_e \in L^2(0,T;W^{1,2}(\Omega))$. Next, we show that $\bar{p}_e$ has vanishing trace at the boundary of Ω. Taking $\mathbf{v} \in H(\mathrm{div};\Omega)$ and applying (21) and (25), one has

$$(\nabla\bar{p}_e, \mathbf{v}) = -(k_o^{-1}(s_e)\bar{\mathbf{q}}_e, \mathbf{v}) = -(\bar{p}_e, \nabla\cdot\mathbf{v}).$$

By using Green theorem for any $\mathbf{v} \in H(\mathrm{div};\Omega)$, we have

$$\int_{\partial\Omega} \bar{p}_e \mathbf{v}\cdot\mathbf{n}\, d\gamma = \int_\Omega \mathbf{v}\cdot\nabla\bar{p}_e + \int_\Omega \bar{p}_e \nabla\cdot\mathbf{v} = 0.$$

It follows that $\bar{p}_e \in L^2(0,T;W_0^{1,2}(\Omega))$, which implies $\bar{p} \in L^2(0,T;W_0^{1,2}(\Omega))$. By the same arguments, we have $p \in L^2(0,T;W_0^{1,2}(\Omega))$.

Similarly, taking $w \in W_0^{1,2}(\Omega) \subset L^2(\Omega)$ in (20) and using (21), one gets that $\bar{p}$ satisfies (17). $\bar{p}$ and p satisfying (19) can be obtained directly from (24).

We now proceed to the time discretization for Problem 1, which is achieved by the Euler implicit scheme. For a given $n \in \{1, 2, ..., N\}$, we define the time discrete mixed variational problem at time t^n. We will show the existence and uniqueness of the solution to the time discrete mixed variational problem. To do so, first we give the discrete form of Problem P.
Problem P^n Find $(s^n, \bar{p}^n, p^n) \in L^2(\Omega) \times W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$, with $s^{n-1} \in L^2(\Omega)$ such that

$$(\frac{s^n - s^{n-1}}{\Delta t}, \phi) + (k_o(s^n)\nabla \bar{p}^n, \nabla \phi) = 0, \tag{26}$$

$$-(\frac{s^n - s^{n-1}}{\Delta t}, \psi) + (k_w(s^n)\nabla p^n, \nabla \psi) = 0, \tag{27}$$

$$(\bar{p}^n - p^n, \lambda) = (p_c(s^n), \lambda) + \tau(\frac{s^n - s^{n-1}}{\Delta t}, \lambda), \tag{28}$$

for all $\phi, \psi \in W_0^{1,2}(\Omega)$ and $\lambda \in L^2(\Omega)$.
Problem 2 (*Semi-discrete Variational Formulation*). Find $(s_e^n, \bar{\mathbf{q}}_e^n, \bar{p}_e^n, \mathbf{q}_e^n, p_e^n) \in L^2(\Omega) \times H(\text{div}; \Omega) \times L^2(\Omega) \times H(\text{div}; \Omega) \times L^2(\Omega)$, with $s_e^{n-1} \in L^2(\Omega)$ such that

$$(\frac{s_e^n - s_e^{n-1}}{\Delta t}, w) + (\nabla \cdot \bar{\mathbf{q}}_e^n, w) = 0, \tag{29}$$

$$(k_o^{-1}(s_e^n)\mathbf{q}_e^n, \mathbf{v}) - (\bar{p}_e^n, \nabla \cdot \mathbf{v}) = 0, \tag{30}$$

$$-(\frac{s_e^n - s_e^{n-1}}{\Delta t}, w) + (\nabla \cdot \mathbf{q}_e^n, w) = 0, \tag{31}$$

$$(k_w^{-1}(s_e^n)\mathbf{q}_e^n, \mathbf{v}) - (p_e^n, \nabla \cdot \mathbf{v}) = 0, \tag{32}$$

$$(\bar{p}_e^n - p_e^n, w) = (p_c(s_e^n), w) + \tau(\frac{s_e^n - s_e^{n-1}}{\Delta t}, w), \tag{33}$$

for all $w \in L^2(\Omega)$ and $\mathbf{v} \in H(\text{div}; \Omega)$.

Similar to Proposition 1 we have the following proposition:

Proposition 2 *Let $n \in \mathbb{N}, n \geq 1$, $s^{n-1} \in L^2(\Omega)$ be given. If $(s^n, \bar{p}^n, p^n) \in L^2(\Omega) \times W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$ is a solution of Problem P^n, then a solution to Problem 2 is given by $s_e^n := s^n$, $\bar{p}_e^n := \bar{p}^n$, $\bar{\mathbf{q}}_e^n = -k_o(s_e^n)\nabla \bar{p}_e^n$, $p_e^n := p^n$, $\mathbf{q}_e^n = -k_w(s_e^n)\nabla \bar{p}_e^n$. Conversely, if $(s_e^n, \bar{\mathbf{q}}_e^n, \bar{p}_e^n, \mathbf{q}_e^n, p_e^n)$ solves Problem 2, then $(s^n, \bar{p}^n, p^n) := (s_e^n, \bar{p}_e^n, p_e^n)$ is a solution of Problem P^n.*

Proof The proof of Proposition 2 is similar to the proof of Proposition 1 (see also [50,51]). We therefore omit it here.

From [12], and Proposition 2 the existence of a solution to Problem 2 is obtained. However, the uniqueness is left open. This is present below

Lemma 1 *Let $n \in \mathbb{N}$, $n \geq 1$ be fixed. Assume that* **(A2)**-**(A4)** *and* **(A6)** *holds. For the time step Δt small enough, Problem 2 has at most one solution.*

Proof Assuming there are two solutions to Problem 2 denoted by $(s_{e1}^n, \bar{\mathbf{q}}_{e1}^n, \bar{p}_{e1}^n, \mathbf{q}_{e1}^n, p_{e1}^n)$ and $(s_{e2}^n, \bar{\mathbf{q}}_{e2}^n, \bar{p}_{e2}^n, \mathbf{q}_{e2}^n, p_{e2}^n)$ for a given $s_e^{n-1} \in L^2(\Omega)$, one has

$$(s_{e1}^n - s_{e2}^n, w) + \Delta t\big(\nabla \cdot (\bar{\mathbf{q}}_{e1}^n - \bar{\mathbf{q}}_{e2}^n), w\big) = 0, \tag{34}$$

$$(k_o^{-1}(s_{e1}^n)\bar{\mathbf{q}}_{e1}^n - k_o^{-1}(s_{e2}^n)\bar{\mathbf{q}}_{e2}^n, \mathbf{v}) - (\bar{p}_{e1}^n - \bar{p}_{e2}^n, \nabla \cdot \mathbf{v}) = 0, \tag{35}$$

$$-(s_{e1}^n - s_{e2}^n, w) + \Delta t\big(\nabla \cdot (\mathbf{q}_{e1}^n - \mathbf{q}_{e2}^n), w\big) = 0, \tag{36}$$

$$(k_w^{-1}(s_{e1}^n)\mathbf{q}_{e1}^n - k_w^{-1}(s_{e2}^n)\mathbf{q}_{e2}^n, \mathbf{v}) - (p_{e1}^n - p_{e2}^n, \nabla\cdot\mathbf{v}) = 0, \tag{37}$$

$$(\bar{p}_{e1}^n - \bar{p}_{e2}^n - p_{e1}^n + p_{e2}^n, w) = \big(p_c(s_{e1}^n) - p_c(s_{e2}^n), w\big) + \frac{\tau}{\Delta t}(s_{e1}^n - s_{e2}^n, w) = 0, \tag{38}$$

for any $w \in L^2(\Omega)$ and $\mathbf{v} \in H(\text{div};\Omega)$.

First, we set $w = \bar{p}_{e1}^n - \bar{p}_{e2}^n$ in (34) and $\mathbf{v} = \Delta t(\bar{\mathbf{q}}_{e1}^n - \bar{\mathbf{q}}_{e2}^n)$ in (35). Adding the results gives

$$(s_{e1}^n - s_{e2}^n, \bar{p}_{e1}^n - \bar{p}_{e2}^n) + \Delta t(k_o^{-1}(s_{e1}^n)\bar{\mathbf{q}}_{e1}^n - k_o^{-1}(s_{e2}^n)\bar{\mathbf{q}}_{e2}^n, \bar{\mathbf{q}}_{e1}^n - \bar{\mathbf{q}}_{e2}^n) = 0. \tag{39}$$

Similarly, taking $w = p_{e1}^n - p_{e2}^n$ in (36) and $\mathbf{v} = \Delta t(\mathbf{q}_{e1}^n - \mathbf{q}_{e2}^n)$ in (37) and adding the results yields

$$-(s_{e1}^n - s_{e2}^n, p_{e1}^n - p_{e2}^n) + \Delta t(k_w^{-1}(s_{e1}^n)\mathbf{q}_{e1}^n - k_w^{-1}(s_{e2}^n)\mathbf{q}_{e2}^n, \mathbf{q}_{e1}^n - \mathbf{q}_{e2}^n) = 0. \tag{40}$$

Adding (39) and (40), we have

$$\begin{aligned}&(\bar{p}_{e1}^n - \bar{p}_{e2}^n - p_{e1}^n + p_{e2}^n, s_{e1}^n - s_{e2}^n) + \Delta t(k_o^{-1}(s_{e1}^n)\bar{\mathbf{q}}_{e1}^n - k_o^{-1}(s_{e2}^n)\bar{\mathbf{q}}_{e2}^n, \bar{\mathbf{q}}_{e1}^n - \bar{\mathbf{q}}_{e2}^n)\\ &+\Delta t(k_w^{-1}(s_{e1}^n)\mathbf{q}_{e1}^n - k_w^{-1}(s_{e2}^n)\mathbf{q}_{e2}^n, \mathbf{q}_{e1}^n - \mathbf{q}_{e2}^n) = 0.\end{aligned} \tag{41}$$

Take $w = s_{e1}^n - s_{e2}^n$ in (38), one obtains

$$(\bar{p}_{e1}^n - \bar{p}_{e2}^n - p_{e1}^n + p_{e2}^n, s_{e1}^n - s_{e2}^n) = \big(p_c(s_{e1}^n) - p_c(s_{e2}^n), s_{e1}^n - s_{e2}^n\big) + \frac{\tau}{\Delta t}\|s_{e1}^n - s_{e2}^n\|^2. \tag{42}$$

Substituting (42) into (41) gives

$$\begin{aligned}&\frac{\tau}{\Delta t}\|s_{e1}^n - s_{e2}^n\|^2 + \big(p_c(s_{e1}^n) - p_c(s_{e2}^n), s_{e1}^n - s_{e2}^n\big)\\ &+\Delta t(k_o^{-1}(s_{e1}^n)\bar{\mathbf{q}}_{e1}^n - k_o^{-1}(s_{e2}^n)\bar{\mathbf{q}}_{e2}^n, \bar{\mathbf{q}}_{e1}^n - \bar{\mathbf{q}}_{e2}^n) + \Delta t(k_w^{-1}(s_{e1}^n)\mathbf{q}_{e1}^n - k_w^{-1}(s_{e2}^n)\mathbf{q}_{e2}^n, \mathbf{q}_{e1}^n - \mathbf{q}_{e2}^n) = 0.\end{aligned} \tag{43}$$

Using **(A2)** and **(A3)**, (43) gives

$$\begin{aligned}&\frac{\tau}{\Delta t}\|s_{e1}^n - s_{e2}^n\|^2 + m_p\|s_{e1}^n - s_{e2}^n\|^2 + \frac{\Delta t}{M_o}\|\bar{\mathbf{q}}_{e1}^n - \bar{\mathbf{q}}_{e2}^n\|^2 + \frac{\Delta t}{M_w}\|\mathbf{q}_{e1}^n - \mathbf{q}_{e2}^n\|^2\\ &\le -\Delta t\Big((k_o^{-1}(s_{e1}^n) - k_o^{-1}(s_{e2}^n))\bar{\mathbf{q}}_{e1}^n, \bar{\mathbf{q}}_{e1}^n - \bar{\mathbf{q}}_{e2}^n\Big) - \Delta t\Big((k_w^{-1}(s_{e1}^n) - k_w^{-1}(s_{e2}^n))\mathbf{q}_{e1}^n, \mathbf{q}_{e1}^n - \mathbf{q}_{e2}^n\Big).\end{aligned}$$

Using **(A6)**, the first term of the right hand side can be estimated as

$$\begin{aligned}&-\Delta t\Big((k_o^{-1}(s_{e1}^n) - k_o^{-1}(s_{e2}^n))\bar{\mathbf{q}}_{e1}^n, \bar{\mathbf{q}}_{e1}^n - \bar{\mathbf{q}}_{e2}^n\Big) \le \Delta t\|\bar{\mathbf{q}}_{e1}^n\|_{L^\infty((0,T)\times\Omega)}\Big(|k_o^{-1}(s_{e1}^n) - k_o^{-1}(s_{e2}^n)|, |\bar{\mathbf{q}}_{e1}^n - \bar{\mathbf{q}}_{e2}^n|\Big)\\ &\le \frac{\Delta t}{2}M_o\|\bar{\mathbf{q}}_{e1}^n|^2_{L^\infty((0,T)\times\Omega)}\|k_o^{-1}(s_{e1}^n) - k_o^{-1}(s_{e2}^n)\|^2 + \frac{\Delta t}{2M_o}\|\bar{\mathbf{q}}_{e1}^n - \bar{\mathbf{q}}_{e2}^n\|^2\\ &\le \frac{\Delta t M_o L_{k_o}^2}{2\delta_o^4}\|\bar{\mathbf{q}}_{e1}^n\|^2_{L^\infty((0,T)\times\Omega)}\|s_{e1}^n - s_{e2}^n\|^2 + \frac{\Delta t}{2M_o}\|\bar{\mathbf{q}}_{e1}^n - \bar{\mathbf{q}}_{e2}^n\|^2.\end{aligned}$$

Here L_{k_o} is the Lipschitz constant of k_o and Young's inequality (see (49)) has been used. A similar estimate can be obtained for the term with k_w. Hence, a constant $C > 0$ exists such that

$$(\frac{\tau}{\Delta t} + m_p - C\Delta t)\|s_{e1}^n - s_{e2}^n\|^2 + \frac{\Delta t}{2M_o}\|\bar{\mathbf{q}}_{e1}^n - \bar{\mathbf{q}}_{e2}^n\|^2 + \frac{\Delta t}{2M_w}\|\mathbf{q}_{e1}^n - \mathbf{q}_{e2}^n\|^2 \le 0,$$

which implies that $s_{e1}^n = s_{e2}^n$, $\bar{\mathbf{q}}_{e1}^n = \bar{\mathbf{q}}_{e2}^n$ and $\mathbf{q}_{e1}^n = \mathbf{q}_{e2}^n$ a.e. for Δt sufficiently small. Substitution of this into (35) and (37) concludes the proof.

For the ease of presentation, in the following we omit the subscript in (20) - (24). The fully discrete mixed scheme for the system is given by the following discrete variational formulation:
Problem 3 (*Discrete Variational Formulation*). Let $n \in \{1, ..., N\}$ and $s_h^{n-1} \in W_h$ be given. Find $(s_h^n, \bar{\mathbf{q}}_h^n, \bar{p}_h^n, \mathbf{q}_h^n, p_h^n) \in W_h \times V_h \times W_h \times V_h \times W_h$ such that

$$(\frac{s_h^n - s_h^{n-1}}{\Delta t}, w_h) + (\nabla \cdot \bar{\mathbf{q}}_h^n, w_h) = 0, \tag{44}$$

$$(k_o^{-1}(s_h^n)\bar{\mathbf{q}}_h^n, \mathbf{v}_h) - (\bar{p}_h^n, \nabla \cdot \mathbf{v}_h) = 0, \tag{45}$$

$$-(\frac{s_h^n - s_h^{n-1}}{\Delta t}, w_h) + (\nabla \cdot \mathbf{q}_h^n, w_h) = 0, \tag{46}$$

$$(k_w^{-1}(s_h^n)\mathbf{q}_h^n, \mathbf{v}_h) - (p_h^n, \nabla \cdot \mathbf{v}_h) = 0, \tag{47}$$

$$(\bar{p}_h^n - p_h^n, w_h) = (p_c(s_h^n), w_h) + \tau(\frac{s_h^n - s_h^{n-1}}{\Delta t}, w_h), \tag{48}$$

for all $w_h \in W_h$ and $\mathbf{v}_h \in V_h$. We take at time $t = 0$: $s_h^0 = P_h s^0$.

Proof The existence of the solution to Problem 3 is a consequence of 2 in [12], while the uniqueness of the solution can be proved as in Lemma 1 above.

3.2 Error estimates

In the following analysis, we will use the Young's inequality

$$ab \le \frac{1}{2\delta}a^2 + \frac{\delta}{2}b^2, \text{ for any } a, b \in \mathbb{R} \text{ and } \delta > 0. \tag{49}$$

We use the following identity, valid for any two families of real vectors $\mathbf{a}^k, \mathbf{b}^k \in \mathbb{R}^m$ $(m \ge 1)$

$$\sum_{k=1}^{N} (\mathbf{a}^k - \mathbf{a}^{k-1}, \mathbf{a}^k) = \frac{1}{2}(|\mathbf{a}^N|^2 - |\mathbf{a}^0|^2 + \sum_{k=1}^{N} |\mathbf{a}^k - \mathbf{a}^{k-1}|^2). \tag{50}$$

Furthermore, we also will use the discrete version of Gronwall's inequality (see [46])

Lemma 2 ***Discrete Gronwall inequality:*** *If $\{y_n\}$, $\{f_n\}$ and $\{g_n\}$ are nonnegative sequences and*

$$y_n \le f_n + \sum_{0 \le k < n} g_k y_k, \quad \text{for all } n \ge 0,$$

then

$$y_n \le f_n + \sum_{0 \le k < n} f_k g_k \exp(\sum_{k < j < n} g_j), \quad \text{for all } n \ge 0.$$

Finally, recall the following result from [54]

Lemma 3 *If the domain Ω is convex, for any $f_h \in W_h$, a $\mathbf{v}_h \in V_h$ exists such that*

$$\nabla \cdot \mathbf{v}_h = f_h, \quad \text{and} \quad \|\mathbf{v}_h\| \le C_\Omega \|f_h\|, \tag{51}$$

where the constant C_Ω does not depend on f_h.

Now we give the convergence result, base on error estimates. To this aim we use the notations

$$e_s^n = s(t^n) - s_h^n, \quad \mathbf{e}_{\bar{\mathbf{q}}}^n = \bar{\mathbf{q}}(t^n) - \bar{\mathbf{q}}_h^n, \quad \mathbf{e}_{\mathbf{q}}^n = \mathbf{q}(t^n) - \mathbf{q}_h^n, \quad e_{\bar{p}}^n = \bar{p}(t^n) - \bar{p}_h^n, \quad e_p^n = p(t^n) - p_h^n.$$

Theorem 1 *Let $(s, \bar{\mathbf{q}}, \bar{p}, \mathbf{q}, p)$ solve Problem 1 and $(s_h^n, \bar{\mathbf{q}}_h^n, \bar{p}_h^n, \mathbf{q}_h^n, p_h^n)$, $n \in \{1, ..., N\}$ solve Problem 3. Assuming that* ***(A1)*** *-* ***(A6)*** *hold, with Δt small enough we have for any $n \in \{1, ..., N\}$,*

$$\|e^n\|^2 := \|e_s^n\|^2 + \|\mathbf{e}_{\bar{\mathbf{q}}}^n\|^2 + \|\mathbf{e}_{\mathbf{q}}^n\|^2 + \|e_{\bar{p}}^n\|^2 + \|e_p^n\|^2 \leq C(\Delta t^2 + h^2),$$

with the constant C not depending on Δt or h.

Proof By subtracting (20) - (24) from (44) - (48), respectively, we obtain

$$\left(\partial_t s - \frac{s_h^n - s_h^{n-1}}{\Delta t}, w_h\right) + \left(\nabla \cdot (\bar{\mathbf{q}}(t^n) - \bar{\mathbf{q}}_h^n), w_h\right) = 0, \tag{52}$$

$$\left(k_o^{-1}(s(t^n))\bar{\mathbf{q}}(t^n) - k_o^{-1}(s_h^n)\bar{\mathbf{q}}_h^n, \mathbf{v}_h\right) - \left(\bar{p}(t^n) - \bar{p}_h^n, \nabla \cdot \mathbf{v}_h\right) = 0, \tag{53}$$

$$-\left(\partial_t s - \frac{s_h^n - s_h^{n-1}}{\Delta t}, w_h\right) + \left(\nabla \cdot (\mathbf{q}(t^n) - \mathbf{q}_h^n), w_h\right) = 0, \tag{54}$$

$$\left(k_w^{-1}(s(t^n))\mathbf{q}(t^n) - k_w^{-1}(s_h^n)\mathbf{q}_h^n, \mathbf{v}_h\right) - \left(p(t^n) - p_h^n, \nabla \cdot \mathbf{v}_h\right) = 0, \tag{55}$$

$$\left(\bar{p}(t^n) - \bar{p}_h^n - p(t^n) + p_h^n, w_h\right) = \left(p_c(s(t^n)) - p_c(s_h^n), w_h\right) + \tau\left(\partial_t s - \frac{s_h^n - s_h^{n-1}}{\Delta t}, w_h\right), \tag{56}$$

for all $w_h \in W_h$ and $\mathbf{v}_h \in V_h$. Using the properties of the projectors Π_h and P_h, the above equations become

$$\left(\partial_t s - \frac{s_h^n - s_h^{n-1}}{\Delta t}, w_h\right) + \left(\nabla \cdot (\Pi_h\bar{\mathbf{q}}(t^n) - \bar{\mathbf{q}}_h^n), w_h\right) = 0, \tag{57}$$

$$\left(k_o^{-1}(s(t^n))\bar{\mathbf{q}}(t^n) - k_o^{-1}(s_h^n)\bar{\mathbf{q}}_h^n, \mathbf{v}_h\right) - \left(P_h\bar{p}(t^n) - \bar{p}_h^n, \nabla \cdot \mathbf{v}_h\right) = 0, \tag{58}$$

$$-\left(\partial_t s - \frac{s_h^n - s_h^{n-1}}{\Delta t}, w_h\right) + \left(\nabla \cdot (\Pi_h\mathbf{q}(t^n) - \mathbf{q}_h^n), w_h\right) = 0, \tag{59}$$

$$\left(k_w^{-1}(s(t^n))\mathbf{q}(t^n) - k_w^{-1}(s_h^n)\mathbf{q}_h^n, \mathbf{v}_h\right) - \left(P_h p(t^n) - p_h^n, \nabla \cdot \mathbf{v}_h\right) = 0, \tag{60}$$

$$\left(P_h\bar{p}(t^n) - \bar{p}_h^n - P_h p(t^n) + p_h^n, w_h\right) = \left(p_c(s(t^n)) - p_c(s_h^n), w_h\right) + \tau\left(\partial_t s - \frac{s_h^n - s_h^{n-1}}{\Delta t}, w_h\right). \tag{61}$$

Taking $w_h = P_h\bar{p}(t^n) - \bar{p}_h^n$ in (57) and $\mathbf{v}_h = \Pi_h\bar{\mathbf{q}}(t^n) - \bar{\mathbf{q}}_h^n$ in (58), adding the results, we obtain

$$\left(\partial_t s - \frac{s_h^n - s_h^{n-1}}{\Delta t}, P_h\bar{p}(t^n) - \bar{p}_h^n\right) + \left(k_o^{-1}(s(t^n))\bar{\mathbf{q}}(t^n) - k_o^{-1}(s_h^n)\bar{\mathbf{q}}_h^n, \Pi_h\bar{\mathbf{q}}(t^n) - \bar{\mathbf{q}}_h^n\right) = 0. \tag{62}$$

Similarly, setting $w_h = P_h p(t^n) - p_h^n$ in (59), and $\mathbf{v}_h = \Pi_h\mathbf{q}(t^n) - \mathbf{q}_h^n$ in (60) we have

$$-\left(\partial_t s - \frac{s_h^n - s_h^{n-1}}{\Delta t}, P_h p(t^n) - p_h^n\right) + \left(k_w^{-1}(s(t^n))\mathbf{q}(t^n) - k_w^{-1}(s_h^n)\mathbf{q}_h^n, \Pi_h\mathbf{q}(t^n) - \mathbf{q}_h^n\right) = 0. \tag{63}$$

Adding (62) and (63) gives

$$\begin{aligned}&\left(\partial_t s-\frac{s_h^n-s_h^{n-1}}{\Delta t}, P_h\bar{p}(t^n)-\bar{p}_h^n-P_hp(t^n)+p_h^n\right)\\&+\Big(k_o^{-1}(s(t^n))\bar{\mathbf{q}}(t^n)-k_o^{-1}(s_h^n)\bar{\mathbf{q}}_h^n, \varPi_h\bar{\mathbf{q}}(t^n)-\bar{\mathbf{q}}_h^n\Big)\\&+\Big(k_w^{-1}(s(t^n))\mathbf{q}(t^n)-k_w^{-1}(s_h^n)\mathbf{q}_h^n, \varPi_h\mathbf{q}(t^n)-\mathbf{q}_h^n\Big)=0. \end{aligned}\tag{64}$$

Taking $w_h=P_h\bar{p}(t^n)-\bar{p}_h^n-P_hp(t^n)+p_h^n$ in (61), one has

$$\begin{aligned}&\left(\partial_t s-\frac{s_h^n-s_h^{n-1}}{\Delta t}, P_h\bar{p}(t^n)-\bar{p}_h^n-P_hp(t^n)+p_h^n\right)\\=&\frac{1}{\tau}\|P_h\bar{p}(t^n)-\bar{p}_h^n-P_hp(t^n)+p_h^n\|^2\\-&\frac{1}{\tau}\Big(p_c(s(t^n))-p_c(s_h^n), P_h\bar{p}(t^n)-\bar{p}_h^n-P_hp(t^n)+p_h^n\Big).\end{aligned}\tag{65}$$

This gives

$$\begin{aligned}&\frac{1}{\tau}\|P_h\bar{p}(t^n)-\bar{p}_h^n-P_hp(t^n)+p_h^n\|^2-\frac{1}{\tau}\Big(p_c(s(t^n))-p_c(s_h^n), P_h\bar{p}(t^n)-\bar{p}_h^n-P_hp(t^n)+p_h^n\Big)\\&+\Big(k_o^{-1}(s(t^n))\bar{\mathbf{q}}(t^n)-k_o^{-1}(s_h^n)\bar{\mathbf{q}}_h^n, \varPi_h\bar{\mathbf{q}}(t^n)-\bar{\mathbf{q}}_h^n\Big)\\&+\Big(k_w^{-1}(s(t^n))\mathbf{q}(t^n)-k_w^{-1}(s_h^n)\mathbf{q}_h^n, \varPi_h\mathbf{q}(t^n)-\mathbf{q}_h^n\Big)=0.\end{aligned}\tag{66}$$

Then we have by **(A3)**

$$\begin{aligned}&\frac{1}{2\tau}\|P_h\bar{p}(t^n)-\bar{p}_h^n-P_hp(t^n)+p_h^n\|^2+\Big(k_o^{-1}(s(t^n))\bar{\mathbf{q}}(t^n)-k_o^{-1}(s_h^n)\bar{\mathbf{q}}_h^n, \varPi_h\bar{\mathbf{q}}(t^n)-\bar{\mathbf{q}}_h^n\Big)\\&+\Big(k_w^{-1}(s(t^n))\mathbf{q}(t^n)-k_w^{-1}(s_h^n)\mathbf{q}_h^n, \varPi_h\mathbf{q}(t^n)-\mathbf{q}_h^n\Big)\le C\|s(t^n)-s_h^n\|^2.\end{aligned}\tag{67}$$

For (67), doing some manipulations, we have

$$\begin{aligned}&\frac{1}{2\tau}\|P_h\bar{p}(t^n)-\bar{p}_h^n-P_hp(t^n)+p_h^n\|^2\\&+\Big(\big(k_o^{-1}(s(t^n))-k_o^{-1}(s_h^n)\big)\bar{\mathbf{q}}(t^n), \mathbf{e}_{\bar{\mathbf{q}}}^n\Big)+\Big(\big(k_o^{-1}(s(t^n))-k_o^{-1}(s_h^n)\big)\bar{\mathbf{q}}(t^n), \varPi_h\bar{\mathbf{q}}(t^n)-\bar{\mathbf{q}}(t^n)\Big)\\&+\Big(k_o^{-1}(s_h^n)\mathbf{e}_{\bar{\mathbf{q}}}^n, \mathbf{e}_{\bar{\mathbf{q}}}^n\Big)+\Big(k_o^{-1}(s_h^n)\mathbf{e}_{\bar{\mathbf{q}}}^n, \varPi_h\bar{\mathbf{q}}(t^n)-\bar{\mathbf{q}}(t^n)\Big)\\&+\Big(\big(k_w^{-1}(s(t^n))-k_w^{-1}(s_h^n)\big)\mathbf{q}(t^n), \mathbf{e}_{\mathbf{q}}^n\Big)+\Big(\big(k_w^{-1}(s(t^n))-k_w^{-1}(s_h^n)\big)\mathbf{q}(t^n), \varPi_h\mathbf{q}(t^n)-\mathbf{q}(t^n)\Big)\\&+\Big(k_w^{-1}(s_h^n)\mathbf{e}_{\mathbf{q}}^n, \mathbf{e}_{\mathbf{q}}^n\Big)+\Big(k_w^{-1}(s_h^n)\mathbf{e}_{\mathbf{q}}^n, \varPi_h\mathbf{q}(t^n)-\mathbf{q}(t^n)\Big)\le C\|e_s^n\|^2.\end{aligned}$$

We name the terms T_1, T_2, T_3, T_4, T_5, T_5, T_6, T_7, T_8, T_9, T_{10} from left to right. Obviously, $T_1\ge 0$. Then by using (49), Cauchy-Schwarz inequality, **(A2)** and **(A6)**, we obtain similar to the proof of Lemma 1,

$$|T_2|=\left|\Big((k_o^{-1}(s(t^n))-k_o^{-1}(s_h^n))\bar{\mathbf{q}}(t^n), \mathbf{e}_{\bar{\mathbf{q}}}^n\Big)\right|\le C\|e_s^n\|^2+\frac{\delta_2}{2}\|\mathbf{e}_{\bar{\mathbf{q}}}^n\|^2,$$

for any $\delta_2 > 0$. Similarly, for T_3 there holds

$$\begin{aligned}|T_3| &= \Big|\Big(\big(k_o^{-1}(s(t^n)) - k_o^{-1}(s_h^n)\big)\bar{\mathbf{q}}(t^n), \Pi_h\bar{\mathbf{q}}(t^n) - \bar{\mathbf{q}}(t^n)\Big)\Big| \\ &\leq C(\|e_s^n\|^2 + \|\bar{\mathbf{q}}(t^n) - \Pi_h\bar{\mathbf{q}}(t^n)\|^2).\end{aligned}$$

Then using the boundedness of the hydraulic conductivity k_o, we get for T_4

$$T_4 = (k_o^{-1}(s_h^n)\mathbf{e}_{\bar{\mathbf{q}}}^n, \mathbf{e}_{\bar{\mathbf{q}}}^n) \geq \frac{1}{M_o}\|\mathbf{e}_{\bar{\mathbf{q}}}^n\|^2.$$

For the term T_5 we use the boundedness of the hydraulic conductivity k_o and the inequality of (49) and Cauchy-Schwarz inequality

$$\begin{aligned}|T_5| &= \Big|\Big(k_o^{-1}(s_h^n)\mathbf{e}_{\bar{\mathbf{q}}}^n, \Pi_h\bar{\mathbf{q}}(t^n) - \bar{\mathbf{q}}(t^n)\Big)\Big| \\ &\leq \frac{\delta_5}{2}\|\mathbf{e}_{\bar{\mathbf{q}}}^n\|^2 + C\|\bar{\mathbf{q}}(t^n) - \Pi_h\bar{\mathbf{q}}^n\|^2,\end{aligned} \tag{68}$$

for any $\delta_5 > 0$. Similarly, we estimate the terms T_6 , T_7, T_8 and T_9 as

$$|T_6| = \Big|\Big((k_w^{-1}(s(t^n)) - k_w^{-1}(s_h^n))\bar{\mathbf{q}}(t^n), \mathbf{e}_{\bar{\mathbf{q}}}^n\Big)\Big| \leq C\|e_s^n\|^2 + \frac{\delta_6}{2}\sum_{n=1}^{N^*}\|\mathbf{e}_{\bar{\mathbf{q}}}^n\|^2,$$

$$|T_7| = \Big|\Big(\big(k_w^{-1}(s(t^n)) - k_w^{-1}(s_h^n)\big)\mathbf{q}(t^n), \Pi_h\mathbf{q}(t^n) - \mathbf{q}(t^n)\Big)\Big| \leq C(\|e_s^n\|^2 + \|\mathbf{q}(t^n) - \Pi_h\mathbf{q}(t^n)\|^2),$$

$$T_8 = (k_w^{-1}(s_h^n)\mathbf{e}_{\mathbf{q}}^n, \mathbf{e}_{\mathbf{q}}^n) \geq \frac{1}{M_w}\|\mathbf{e}_{\mathbf{q}}^n\|^2,$$

$$|T_9| = \Big|\Big(k_w^{-1}(s_h^n)\mathbf{e}_{\mathbf{q}}^n, \Pi_h\mathbf{q}(t^n) - \mathbf{q}(t^n)\Big)\Big| \leq \frac{\delta_9}{2}\|\mathbf{e}_{\mathbf{q}}^n\|^2 + C\|\mathbf{q}(t^n) - \Pi_h\mathbf{q}_h^n\|^2.$$

We gather T_1 to T_{10} and choose δ_2, δ_5, δ_6 and δ_9 properly to have

$$\|\mathbf{e}_{\bar{\mathbf{q}}}^n\|^2 + \|\mathbf{e}_{\mathbf{q}}^n\|^2 \leq C(\|e_s^n\|^2 + \|\bar{\mathbf{q}}(t^n) - \Pi_h\bar{\mathbf{q}}_h^n\|^2 + \|\mathbf{q}(t^n) - \Pi_h\mathbf{q}_h^n\|^2). \tag{69}$$

Furthermore, taking $\nabla\cdot\mathbf{v}_h = P_h\bar{p}(t^n) - \bar{p}_h^n$ in (53) and applying Lemma 3 we obtain

$$\begin{aligned}\|\bar{p}(t^n) - \bar{p}_h^n\|^2 &= \big(k_o^{-1}(s(t^n))\bar{\mathbf{q}}(t^n) - k_o^{-1}(s_h^n)\bar{\mathbf{q}}_h^n, \mathbf{v}_h\big) - \big(\bar{p}(t^n) - \bar{p}_h^n, P_h\bar{p}(t^n) - \bar{p}(t^n)\big) \\ &\leq C\big(\|e_s^n\| + \|\mathbf{e}_{\bar{\mathbf{q}}}^n\|\big)\|P_h\bar{p}(t^n) - \bar{p}_h^n\| + \|\bar{p}(t^n) - \bar{p}_h^n\|\|P_h\bar{p}(t^n) - \bar{p}(t^n)\| \\ &\leq C\big(\|e_s^n\|^2 + \|\mathbf{e}_{\bar{\mathbf{q}}}^n\|^2\big) + \frac{3}{2}\|P_h\bar{p}(t^n) - \bar{p}(t^n)\|^2 + \frac{3}{4}\|\bar{p}(t^n) - \bar{p}_h^n\|^2,\end{aligned} \tag{70}$$

which gives by applying (15)

$$\|e_{\bar{p}}^n\|^2 = \|\bar{p}(t^n) - \bar{p}_h^n\|^2 \leq C\big(\|e_s^n\|^2 + \|\mathbf{e}_{\bar{\mathbf{q}}}^n\|^2 + h^2\big). \tag{71}$$

Similarly, by taking $\nabla\cdot\mathbf{v}_h = P_hp(t^n) - p_h^n$ in (55), one also has

$$\|e_p^n\|^2 = \|p(t^n) - p_h^n\|^2 \leq C\big(\|e_s^n\|^2 + \|\mathbf{e}_{\mathbf{q}}{}^n\|^2 + h^2\big). \tag{72}$$

Setting $w_h = P_h s(t^n) - s_h^n$ in (61) and summing up results from $n = 1$ to N^*, $N^* \in \{1, ..., N\}$, and doing some manipulations, we have

$$\sum_{n=1}^{N^*}\left(\partial_t s - \frac{s(t^n)-s(t^{n-1})}{\Delta t}, e_s^n\right)$$
$$+\sum_{n=1}^{N^*}\left(\partial_t s - \frac{s_h^n - s_h^{n-1}}{\Delta t}, P_h s(t^n) - s(t^n)\right) + \sum_{n=1}^{N^*}\left(\frac{s(t^n)-s(t^{n-1}) - s_h^n + s_h^{n-1}}{\Delta t}, e_s^n\right)$$
$$=\frac{1}{\tau}\sum_{n=1}^{N^*}\left(P_h\bar{p}(t^n) - \bar{p}_h^n - P_h p(t^n) + p_h^n, P_h s(t^n) - s_h^n\right) - \frac{1}{\tau}\sum_{n=1}^{N^*}\left(p_c(s(t^n)) - p_c(s_h^n), P_h s(t^n) - s_h^n\right).$$

We proceed as before, denoting the terms by S_1, S_2, S_3, S_4, S_5, S_5. Then we have

$$S_1 + S_2 + S_3 = S_4 + S_5.$$

We use the inequalities of (49) and Cauchy-Schwarz to estimate S_1:

$$|S_1| = \left|\sum_{n=1}^{N^*}\left(\partial_t s - \frac{s(t^n)-s(t^{n-1})}{\Delta t}, e_s^n\right)\right| \leq \frac{1}{\Delta t}\sum_{n=1}^{N^*}\|\Delta t\partial_t s - (s(t^n) - s(t^{n-1}))\|\cdot\|e_s^n\|$$
$$\leq \underbrace{\frac{1}{2\Delta t^2}\sum_{n=1}^{N^*}\left\|\int_{t^{n-1}}^{t^n}(\partial_t s(t^n) - \partial_t s(z))dz\right\|^2}_{S_{11}} + \frac{1}{2}\sum_{n=1}^{N^*}\|e_s^n\|^2.$$

Further, by using the Bochner inequality and since $\partial_{tt}s \in L^2(0,T;L^2(\Omega))$ (see [41] Proposition 4.2), we estimate S_{11}

$$\begin{aligned}S_{11} =& \frac{1}{2\Delta t^2}\sum_{n=1}^{N^*}\left\|\int_{t^{n-1}}^{t^n}\int_z^{t^n}\partial_{tt}s(\eta)d\eta dz\right\|^2\\ \leq& \frac{1}{2\Delta t^2}\Delta t^2\sum_{n=1}^{N^*}\left\|\int_{t^{n-1}}^{t^n}\partial_{tt}s(\eta)d\eta\right\|^2\\ \leq& \frac{\Delta t}{2}\sum_{n=1}^{N^*}\int_{t^{n-1}}^{t^n}\|\partial_{tt}s(\eta)\|^2 d\eta = \frac{\Delta t}{2}\int_0^T\|\partial_{tt}s(\eta)\|^2 d\eta\\ \leq& C\Delta t.\end{aligned}$$

For S_2, by using the properties of the projectors (13) and applying (15) one gets

$$\begin{aligned}|S_2| &= \left|\sum_{n=1}^{N^*}\left(\partial_t s - \frac{s_h^n - s_h^{n-1}}{\Delta t}, P_h s(t^n) - s(t^n)\right)\right| = \left|\sum_{n=1}^{N^*}\left(\partial_t s, P_h s(t^n) - s(t^n)\right)\right|\\ &= \left|\sum_{n=1}^{N^*}\left(\partial_t s - P_h\partial_t s, P_h s(t^n) - s(t^n)\right)\right| \leq N^* h^2 \leq \frac{Th^2}{\Delta t}.\end{aligned}$$

Now we estimate the term S_3. By using (50), we have

$$S_3 = \sum_{n=1}^{N^*}\left(\frac{s(t^n)-s(t^{n-1}) - s_h^n + s_h^{n-1}}{\Delta t}, e_s^n\right) = \frac{1}{2}\frac{\|e_s^{N^*}\|^2}{\Delta t} - \frac{1}{2}\frac{\|s^0 - P_h s^0\|^2}{\Delta t} + \frac{1}{2}\sum_{n=1}^{N^*}\|e_s^n - e_s^{n-1}\|^2.$$

For S_4 and S_5 we have

$$
\begin{aligned}
|S_4| &= \left| \frac{1}{\tau} \sum_{n=1}^{N^*} \left(P_h\bar{p}(t^n) - \bar{p}_h^n - P_h p(t^n) + p_h^n, P_h s(t^n) - s(t^n) + s(t^n) - s_h^n \right) \right| \\
&= \left| \frac{1}{\tau} \sum_{n=1}^{N^*} \left(P_h\bar{p}(t^n) - \bar{p}(t^n) + \bar{p}(t^n) - \bar{p}_h^n - P_h p(t^n) + p(t^n) - p(t^n) + p_h^n, s(t^n) - s_h^n \right) \right| \\
&\leq \frac{C}{\tau} \sum_{n=1}^{N^*} \left(\|e_{\bar{p}}^n\|^2 + \|e_p^n\|^2 + \|P_h\bar{p}(t^n) - \bar{p}(t^n)\|^2 + \|P_h p(t^n) - p(t^n)\|^2 + \|e_s^n\|^2 \right),
\end{aligned}
$$

$$
|S_5| \leq \frac{C}{\tau} \left(\sum_{n=1}^{N^*} \|P_h s(t^n) - s(t^n)\|^2 + \sum_{n=1}^{N^*} \|e_s^n\|^2 \right).
$$

We gather the estimates for S_1 to S_6 and apply the properties of the operators P_h and Π_h to get

$$
\|e_s^{N^*}\|^2 \leq C(h^2 + \Delta t^2) + C \sum_{n=1}^{N^*} \Delta t \|e_s^n\|^2 + C \sum_{n=1}^{N^*} \Delta t \|e_{\bar{p}}^n\|^2 + C \sum_{n=1}^{N^*} \Delta t \|e_p^n\|^2.
$$

Applying the estimates from (69), (71) and (72), we have

$$
\|e_s^{N^*}\|^2 \leq C(h^2 + \Delta t^2) + C \sum_{n=1}^{N^*} \Delta t \|e_s^n\|^2.
$$

By using the Gronwall's inequality with Lemma 2, we obtain

$$
\|e_s^{N^*}\|^2 \leq C(h^2 + \Delta t^2). \tag{73}
$$

The proof is completed by substituting (73) into (69), (71), and (72).

4 Numerical results

For the numerical results, triangular meshes from the FVCA8 benchmark mesh repository has been used. The domain selected is the unit square

$$
\Omega = (0, 1) \times (0, 1). \tag{74}
$$

The results presented in Section 3 are for two phase equation without gravity. However it was mentioned in Remark 3 that the analysis also works in the presence of gravity and so for the numerical results we have included gravity terms in (8) and (9): in dimensional form they read

$$
\bar{\mathbf{q}} = -k_o(s)(\nabla \bar{p} - \rho_o \hat{g}), \tag{75}
$$

$$
\mathbf{q} = -k_w(s)(\nabla p - \rho_w \hat{g}), \tag{76}
$$

with $\hat{g}$ being the unit vector along the direction of gravity and ρ_o, ρ_w being the non-dimensionalized densities of the non-wetting (oil) and the wetting (water) phases. $\hat{g}$ has been taken pointing along the x-direction. $\rho_o = 0.8$ and $\rho_w = 1$ are used throughout. Similarly, source terms f_o and f_w have been added in (6) and (7) respectively.

The functions k_o, k_w and p_c used in the computations, are

$$k_o(s) = s, \quad k_w(s) = (1 - s), \quad p_c(s) = s. \tag{77}$$

The source terms f_o, f_w have the form

$$f_o = \frac{e^{-t}}{4}[1 + x^2 + y^2 - 2\rho_o x + \tau((1 + 2x^2 + 2y^2)e^{-t} - 4)], \tag{78}$$

$$f_w = -\frac{e^{-t}}{4}[1 + x^2 + y^2 - 2\rho_w x + (1 + 2x^2 + 2y^2)e^{-t}]. \tag{79}$$

The initial and boundary conditions are shown in Table 1.

Table 1 Assumed initial and boundary conditions.

IC	Ω	
$t = 0$	$s(x, y, 0) = 1 - \frac{1}{4}(1 + x^2 + y^2)$	
BC	Ω	
$x = 0$	$\bar{p}(0, y, t) = 1 + \frac{\tau}{4}(1 + y^2)e^{-t}$	$p(0, y, t) = \frac{1}{4}(1 + y^2)e^{-t}$
$x = 1$	$\bar{p}(1, y, t) = 1 + \frac{\tau}{4}(2 + y^2)e^{-t}$	$p(1, y, t) = \frac{1}{4}(2 + y^2)e^{-t}$
$y = 0$	$\bar{\mathbf{q}}_y = 0$	$\mathbf{q}_y = 0$
$y = 1$	$\bar{\mathbf{q}}_y = -\frac{\tau}{2}(1 - \frac{1}{4}(2 + x^2)e^{-t})e^{-t}$	$\mathbf{q}_y = -\frac{1}{8}(2 + x^2)e^{-2t}$

Observe that both Dirichlet and Neuman boundary conditions have been used instead of the zero Dirichlet condition assumed for the analysis in Section 3. This is to show that the result of Theorem 1 stays valid for more general boundary conditions.

One can directly verify that under these conditions the exact solution of the system is given by,

$$s = 1 - \frac{1}{4}(1 + x^2 + y^2)e^{-t}, \quad \bar{p} = 1 + \frac{\tau}{4}(1 + x^2 + y^2)e^{-t}, \quad p = \frac{1}{4}(1 + x^2 + y^2)e^{-t}, \tag{80}$$

with $\bar{\mathbf{q}}$ and $\mathbf{q}$ calculated from (75) and (76).

Four different mesh sizes have been used for the computations, i.e. $h = 0.1$, 0.05, 0.02 and 0.01 and $\tau = 1$ has been taken throughout. Figure 1 (left) shows the saturation for the exact and the numerical solution at $t = 1$. The two profiles look almost superimposed on each other which implies that the computational results are correct. As the equations are nonlinear, some kind of linearization techniques have to be used. In our computations we have used the L-scheme as presented in [53] for this purpose. Figure 1 (right) shows how the errors decrease with iterations of the linearization step. An error cut-off of 10^{-8} has been used in order to make sure that the errors due to the nonlinear solver do not influence the results of the order estimate.

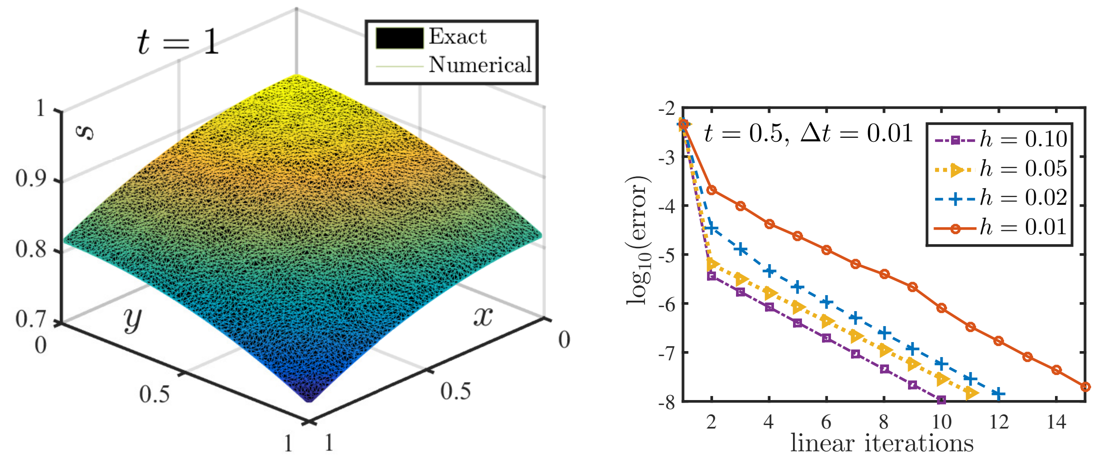

Figure 1 (left) Saturation for the exact and the numerical solution at $t = 1$. Here $h = 0.02$ and $\Delta t = .001$. (right) The error characteristics of the linear iterations. $\Delta t = 0.01$ is used in all the computations. The error is calculated as $(\|p_n^i - p_n^{i-1}\|^2 + \|\bar{p}_n^i - \bar{p}_n^{i-1}\|^2 + \|s_n^i - s_n^{i-1}\|^2)^{1/2}$, where the superscript i indicates the solution at the i^{th} iteration.

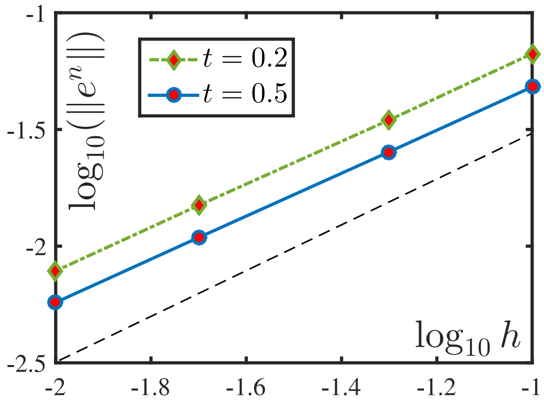

Figure 2 Error $\|e^n\|$ vs. $\log_{10} h$. h and Δt follow the relation (81). The errors are plotted at two times, $t = 0.2, 0.5$. The black dashed line represents the slope of 1.

To recover the order of convergence of the scheme, we vary h and Δt so that

$$h = \Delta t. \tag{81}$$

By taking h in this form one ensures that $\log_{10}\sqrt{(h^2 + \Delta t^2)} = \log_{10} h + \text{constant}$. So if the slope of $\log_{10}\|e^n\|$ against $\log_{10} h$ is 1 then it would support the analytical findings in Section 3. This is precisely the case as shown in Figure 2. It shows the error plots at two times, $t = 0.2, 0.5$. The lines are nearly parallel to the black dashed line, representing slope of 1.

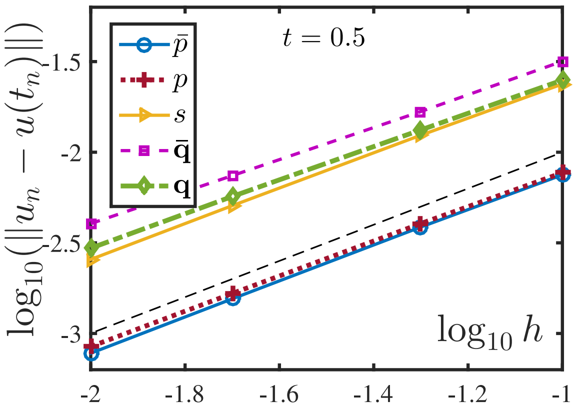

Figure 3 Error $\log_{10}(\|u_n - u(t_n)\|)$ vs. $\log_{10} h$. Here $u = p, \bar{p}, s, \mathbf{q}, \bar{\mathbf{q}}$. All the plots are for $t = 0.5$ and h, Δt are such that $h = \Delta t$. The black dashed line represents the slope of 1.

An interesting observation is that the individual components of $\|e^n\|$ also scales with h when $h = \Delta t$. This is shown in Figure 3. For $h = \Delta t$ the log plots of e_s^n, e_p^n, $e_{\bar{p}}^n$, $e_{\mathbf{q}}^n$, $e_{\bar{\mathbf{q}}}^n$ are all parallel to the black dashed line representing slope of 1. This supports parts of the proof of Theorem 1, e.g. (69), (73).

Next, h and Δt are varied in a general way so that $h = \Delta t$ is not always satisfied. We plot the errors $\|e^n\|^2$ against Δt^2 in Figure 4 (left) and against h^2 in Figure 4 (right). The nearly straight profiles of the errors, for a fixed h and for a fixed Δt, implies that the errors behave as

$$\|e_n\|^2 \approx Ah^2 + B\Delta t^2, \tag{82}$$

where $A, B > 0$ are constants. This is similar to what one expects from the proof of Theorem 1, for small enough Δt. Relation (82) still gives linear profiles in Figure 2. To estimate the values of A and B, the average slopes of the line $h = 0.1$ in Figure 4 (left) and the line $\Delta t = 0.5$ in Figure 4 (right) are calculated. This gives $A = .02$ and $B = .22$.

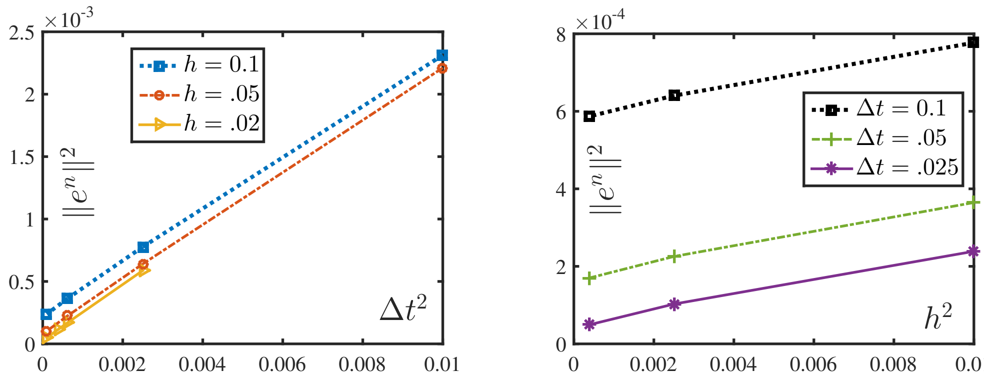

Figure 4 (left) Errors $\|e^n\|^2$ vs. Δt^2 for fixed h. (right) Errors $\|e^n\|^2$ vs. h^2 for fixed Δt. $h = .1$, $.05$, $.02$ and $t = 0.5$ have been used throughout.

We use the values of A and B to predict the error for any Δt and h. This is shown in Table 2. Note that the value of A and B is derived from the line $h = .1$ and $\Delta t = .05$ in Figure 4 but they are used to estimate the rest of the errors. The match is reasonably close.

$h = 0.1$	$\Delta t = 0.1$	$\Delta t = 0.05$	$\Delta t = 0.025$	$\Delta t = 0.01$
$\log_{10}(\|e^n\|)$	-1.3179	-1.5548	-1.7191	-1.8110
$\log_{10}\left(\sqrt{Ah^2 + B\Delta t^2}\right)$	-1.3099	-1.5625	-1.7359	-1.8268
$h = 0.05$	$\Delta t = 0.1$	$\Delta t = 0.05$	$\Delta t = 0.025$	$\tau = 0.01$
$\log_{10}(\|e^n\|)$	-1.3277	-1.5968	-1.8235	-1.9943
$\log_{10}\left(\sqrt{Ah^2 + B\Delta t^2}\right)$	-1.3239	-1.6109	-1.8635	-2.0713
$h = 0.02$	$\Delta t = 0.1$	$\Delta t = 0.05$	$\Delta t = 0.025$	$\Delta t = 0.01$
$\log_{10}(\|e^n\|)$	diverged	-1.6156	-1.8844	-2.1522
$\log_{10}\left(\sqrt{Ah^2 + B\Delta t^2}\right)$	-	-1.6267	-1.9186	-2.2614
$h = 0.01$	$\Delta t = 0.1$	$\Delta t = 0.05$	$\Delta t = 0.025$	$\Delta t = 0.01$
$\log_{10}(\|e^n\|)$	diverged	-1.6218	-1.9075	-2.2413
$\log_{10}\left(\sqrt{Ah^2 + B\Delta t^2}\right)$	-	-1.6290	-1.9277	-2.3099

Table 2 $\log_{10}(\|e^n\|)$ vs. $\log_{10}\left(\sqrt{Ah^2 + B\Delta t^2}\right)$ for different meshsizes and timesteps. $A = 0.02$ and $B = .22$ are used and they are calculated from Figure 4. In all the results, $t = 0.5$.

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References

1. B. Andreianov and C. Cancès, A phase-by-phase upstream scheme that converges to the vanishing capillarity solution for countercurrent two-phase flow in two-rocks media, Comput. Geosci., 18:211-226, 2014.
2. B. Andreianov, R. Eymard, M. Ghilani and N. Marhraoui. Finite volume approximation of degenerate two-phase flow model with unlimited air mobility. Numer. Meth. Part Differ. Equ., 29 (2013), 441-474.
3. T. Arbogast, An error analysis for Galerkin approximations to an equation of mixed elliptic-parabolic type, Technical Report TR90-33, Department of Computational and Applied Mathematics, Rice University, Houston, TX, 1990.
4. T. Arbogast, M. Obeyeskere and M.F. Wheeler, Numerical methods for the simulation of flow in root-soil systems, SIAM J. Numer. Methods Engrg. 45: 1025-1046, 1999.
5. J. Bear, Dynamics of Fluids in Porous Media, Elsevier, New York, 1972.
6. S. Bottero, S.M. Hassanizadeh, P.J. Kleingeld and T.J. Heimovaara, , Nonequilibrium capillarity effects in two-phase flow through porous media at different scales, Water Resour. Res. 47.10 (2011)
7. K. Brenner, C. Cancès and D. Hilhorst, Finite volume approximation for an immiscible two-phase flow in porous media with discontinuous capillary pressure, Comput. Geosci. 17 (2013) 573-597.
8. F. Brezzi and M. Fortin, Mixed and Hybrid Finite Element Methods, Springer-Verlag, New York, 1991.
9. F. Brunner, F. A. Radu and P. Knabner. Analysis of an upwind-mixed hybrid finite element method for transport problems. SIAM J. Numer. Anal., 52(1) (2014) 83-102.
10. C. Cancès, I.S. Pop and M. Vohralík. An a posteriori error estimate for vertex-centered finite volume discretizations of immiscible incompressible two-phase flow. Math. Comput., 83 (2014) 153-188.
11. X. Cao and I.S. Pop. Uniqueness of weak solutions for a pseudo-parabolic equation modeling two phase flow in porous media, Appl. Math. Lett., 46 (2015) 25-30.
12. X. Cao and I.S. Pop. Two-phase porous media flows with dynamic capillary effects and hysteresis: uniqueness of weak solutions, Comput. Math. Appl., 69 (2015) 688-695.

13. X. Cao and I.S. Pop. Degenerate two-phase porous media flow model with dynamic capillarity, J. Differ. Equ., 260 (2016) 2418-2456.
14. X. Cao, S. F. Nemadjieu and S. Pop. Convergence of a MPFA finite volume scheme for a two phase porous media flow model with dynamic capillarity, IMA J. Numer. Anal., accepted.
15. Z. Chen. Degenerate two-phase incompressible flow I. existence, uniqueness and regularity of a weak solution. J. Differ. Equ., 171 (2001) 203-232.
16. D.A. DiCarlo. *Experimental measurements of saturation overshoot on infiltration*, Water Resour. Res. **40**, W04215.1-W04215 (2004).
17. H. Nasser El Dine and M. Saad. Analysis of a finite volume-finite element method for Darcy-Brinkman two-phase flows in porous media. J. Comput. Appl. Math., 337 (2018) 51-72.
18. H. Nasser El Dine, M. Saad and R. Talhouk. A Finite Volume Scheme for Darcy-Brinkman's Model of Two-Phase Flows in Porous Media. DOI10.1007/978-3-319-63082-3-104. In book: Progress in Industrial Mathematics at ECMI 2016.
19. J.Jr. Douglas, R. E. Ewing and M. F. Wheeler. The approximation of the pressure by a mixed method in the simulation of miscible displacement. RAIRO Anal. Numér., 17 (1983) 17-33.
20. J. Droniou, R. Eymard, D. Hilhorst, and X. D. Zhou. Convergence of a finite-volume mixed finite-element method for an elliptic-hyperbolic system. IMA J Numer Anal (2003) 23 (3): 507-538.
21. C.J. van Duijn, X. Cao and I. Pop. Two-phase flow in porous media: dynamic capillarity and heterogeneous media. Transp. Porous Med., 110 (2015) 1-26.
22. C.J. van Duijn, Y. Fan, L.A. Peletier and S. Pop. Travelling wave solutions for degenerate pseudo-parabolic equation modelling two-phase flow in porous media, Nolinear Anal. Real World Appl., Vol. 14 (2013), 1361-1383.
23. C.J. van Duijn, K. Mitra and I.S. Pop. Travelling wave solutions for the Richards equation incorporating non-equilibrium effects in the capillarity pressure, Nolinear Anal. Real World Appl. 41 (2018) 232-268.
24. L.J. Durlofsky. Coarse scale models of two phase flow in heterogeneous reservoirs: volume averaged equations and their relationship to existing upscaling techniques. Comput. Geosci., 2 (1998) 73-92.
25. M.G. Edwards. Unstructured control-volume distributed full tensor finite volume schemes with flow based grids. Comput. Geosci. 6(2002) 433-452.
26. Y. Epshteyn, B. Riviere. Analysis of discontinuous Galerkin methods for incompressible two-phase flow. J. Comput. Appl. Math. 225(2009) 487?509.
27. R. Barrett, M. Berry, T.F. Chan, J. Demmel, J. Donato, J. Dongarra, V. Eijkhout, R. Pozo, C. Romine, H.v.d. Vorst, Templates for the Solution of Linear Systems: Building Blocks for Iterative Methods, (1994).
28. R. E. Ewing and M.F. Wheeler. Galerkin methods for miscible displacement problems with point sources and sinks-unit mobility ratio case, in Mathematical Methods in Energy Research (Laramie, WY, 1982/1983), SIAM, Philadelphia, (1984) 40-58.
29. R. Eymard, R. Herbin and A. Michel. Mathematical study of a petroleum-engineering scheme. ESAIM-Math. Model. Numer. Anal.-Model. Math. Anal. Numer., 37 (2003) 937-972.
30. Y. Fan and S. Pop. A class of pseudo-parabolic equations: existence, uniqueness of weak solutions, and error estimates for the Euler-implicit discretization, Math. Methods Appl. Sci., 34 (2011) 2329-2339.
31. Y. Fan and S. Pop. Equivalent formulations and numerical schemes for a class of pseudo-parabolic equations, J. Comput. Appl. Math., 246 (2013), 86-93.
32. R. Fučík, J. Mikyška, T. Sakaki, M. Beneš and T.H. Illangasekare, Significance of dynamic effect in capillarity during drainage experiments in layered porous media. Vadose Zone J. 9 (2010) 697-708.
33. S. M. Hassanizadeh and W. G. Gray, Thermodynamic basis of capillary pressure in porous media, Water Resour. Res., 29 (1993) 3389-3405.
34. R. Helmig, Multiphase Flow and Transport Processes in the Subsurface: a Contribution to the Modeling of Hydrosystems, Springer, Berlin, 1997.
35. R. Helmig, A. Weiss, and B.I. Wohlmuth. Dynamic capillary effects in heterogeneous porous media. Comput. Geosci., 11 (2007) 261-274.
36. R. Helmig, A. Weiss and B.I. Wohlmuth. Variational inequalities for modeling flow in heterogeneous porous media with entry pressure. Comput. Geosci., 13 (2009) 373-389.
37. S. Karpinski and I.S. Pop. Analysis of an interior penalty discontinuous Galerkin scheme for two phase flow in porous media with dynamic capillarity effects. Numer. Math. 136 (2017), 249-286.
38. S. Karpinski, I.S. Pop, F.A. Radu. Analysis of a linearization scheme for an interior penalty discontinuous Galerkin method for two phase flow in porous media with dynamic capillarity effects. INT. J. Numer. Meth. Eng. 112 (2017) 553-577.
39. J. Koch, A. Rätz and B. Schweizer. Two-phase flow equations with a dynamic capillary pressure. European J. Appl. Math., 24 (2013) 49-75.
40. D. Kroener and S. Luckhaus. Flow of oil and water in a porous medium. J. Differ. Equ., 55 (1984) 276-288.
41. A. Lamacz and A. Rätz and B. Schweizer, A well-posed hysteresis model for flows in porous media and applications to fingering effects, Adv. Math. Sci. Appl. 21 (2011) 33-64.
42. S. Manthey, S.M. Hassanizadeh, R. Helmig, and R. Hilfer, Dimensional analysis of two-phase flow including a rate-dependent capillary pressure-saturation relationship, Adv. in Water Resour. 31.9 (2008): 1137-1150

43. A. Michel. A finite volume scheme for two-phase immiscible flow in porous media. SIAM J. Numer. Anal., 41 (2003) 1301-1317.
44. A. Mikelić, A global existence result for the equations describing unsaturated flow in porous media with dynamic capillary pressure, J. Differ. Equ. 248 (2010) 1561-1577.
45. J.M. Nordbotten and M.A. Celia, Geological storage of CO_2: modeling approaches for large-scale simulation, John Wiley & Sons, New Jersey, 2012.
46. M. Ohlberger. Convergence of a mixed finite elements-finite volume method for the two phase flow in porous media. East-West J. Numer. Math., 5 (1997) 183-210.
47. M. Peszynska and S.-Y. Yi. Numerical methods for unsaturated flow with dynamic capillary pressure in heterogeneous porous media. Int. J. Numer. Anal. Model., 5 (2008) 126-149.
48. I.S. Pop, F. Radu and P. Knabner, Mixed finite elements for the Richard's equation: linearization procedure, J. Comput. Appl. Math. 168 (2004) 365-373.
49. A. Quarteroni and A. Valli, Numerical Approximations of Partial Differential Equations, Springer-Verlag, 1994.
50. F.A. Radu and W. Wang, Convergence analysis for a mixed finite element scheme for flow in strictly unsaturated porous media, Nonlinear Anal.-Real World Appl. 15:266-275, 2014.
51. F.A. Radu, I.S. Pop and P. Knabner. Oder of convergence estimates for an Euler implicit, mixed finite element discretization of Richards' equation. SIAM J. Numer. Anal, 42(4) (2004) 1452-1478.
52. F.A. Radu, I.S. Pop and P. Knabner. Error estimates for a mixed finite element discretization of some degenerate parabolic equations. Numer. Math. 109(2008) 285-311.
53. F.A. Radu, J.M. Nordbotten, I.S. Pop. and K. Kumar. A robust linearization scheme for finite volume based discretizations for simulation of two-phase flow in porous media. J. Comput. Appl. Math. 289 (2015) 134-141.
54. J.M. Thomas, Sur l' analyse numérique des méthodes d'éléments finis hybrides et mixtes, Thése d'Etat, Université Pierre & Marie Curie, Paris 6, 1977.

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